On the Non-Analytic Examples of Christ and Geller

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1. Introduction. In [3], M. Christ and D. Geller gave the following remarkable counterexample to analytic hypoellipticity of $\bar{\partial}_b$ for real analytic CR manifolds of finite type:

Theorem 1.1. *On the three-dimensional CR manifold $M_m := \{\mathrm{Im} z_2 = [\mathrm{Re} z_1]^{2m}\}$ $(m = 2,3,\ldots)$, $\bar{\partial}_b$ fails to be relatively analytic hypoelliptic.*

Here, *relative analytic hypoellipticity* of $\bar{\partial}_b$ is a notion different from the usual one. Its definition is given in §2.

Moreover, relative analytic hypoellipticity of $\bar{\partial}_b$ is closely connected with real analyticity of the Szegö kernel off the diagonal. By considering the Szegö kernel as a singular solution of the equation $\bar{\partial}_b u = 0$, Christ and Geller obtained Theorem 1.1 as a corollary of the following theorem.

Theorem 1.2. *The Szegö kernel of $M_m (m = 2,3,\ldots)$ fails to be real analytic off the diagonal.*

The proof of Theorem 1.2 by Christ and Geller [3] is based on certain formulas of Nagel [5]. Though their proof is logically clear, it seems to be difficult to understand the singularity of the Szegö kernel of M_m directly. On the other hand, M. Christ [1] constructed singular solutions for $\bar{\partial}_b$ directly, and proved Theorem 1.1. In this note, we give the Szegö kernel of M_m an integral representation in terms of the singular solutions of Christ. Since the singular solutions of Christ are substantially simpler, our representation makes it easy to understand the singularity of the Szegö kernel of M_m. We also give a similar representation to the Bergman kernel of the domain $\{\mathrm{Im} z_2 > [\mathrm{Re} z_1]^{2m}\} \subset \boldsymbol{C}^2 (m = 2,3,\ldots)$ on the boundary.

Finally we remark that our subject in this note has also been treated in the paper of M. Christ ([2], §7), and our result can be considered as an improvement of Proposition 7.2 in [2].

2. Statement of our results. Consider the hypersurface $M := \{\mathrm{Im} z_2 = P(z_1)\} \subset \boldsymbol{C}^2$, where $P : \boldsymbol{C} \to \boldsymbol{R}$ is a subharmonic, nonharmonic polynomial. Such a surface is pseudoconvex and of finite type. A nonvanishing, antiholomorphic, tangent vector field is $\partial / \partial \bar{z}_1 - 2i(\partial P / \partial \bar{z}_1) \partial / \partial \bar{z}_2$. As coordinates for the surface we use $\boldsymbol{C} \times \boldsymbol{R} \ni (z = x + iy, t) \mapsto (z, t + iP(z))$; the vector field pulls back to $\bar{\partial}_b = \partial / \partial \bar{z} - i(\partial P / \partial \bar{z}) \partial / \partial t$. Let $\bar{\partial}_b^*$ denote the formal adjoint of $\bar{\partial}_b$ with respect to the Lebesgue measure on $\boldsymbol{C} \times \boldsymbol{R}$. Recall the natural notion of analytic hypoellipticity for $\bar{\partial}_b$:

Definition 2.1. *$\bar{\partial}_b$ is relatively analytic hypoelliptic on M, if whenever $\bar{\partial}_b u$ is real analytic in an open set V and $u = \bar{\partial}^*{}_b v$ for some $v \in L^2$ in V, u is real analytic in V.*

In usual sense, $\bar{\partial}_b$ is not even C^∞ hypoelliptic, but it is well-known that if $\bar{\partial}_b \bar{\partial}_b^* u \in C^\infty$, then $\bar{\partial}_b^* u \in C^\infty$ ([4]).

When $M = M_m = \{\mathrm{Im} z_2 = [\mathrm{Re} z_1]^{2m}\}$ $(m = 2,3,\ldots)$, M. Christ in [1], [2] constructed the singular solutions of the equation $\bar{\partial}_b u = 0 (u = \bar{\partial}_b^* v,\ v \in L^2)$ by applying the partial Fourier transformation and solving a certain simple ordinary differential equation. Christ's solutions are of the following form:

$$S_j^v(z, t) = \int_0^\infty e^{it\tau} e^{-x^{2m}\tau} e^{\sigma(y) i a_j z \tau^{\frac{1}{2m}}} \tau^v d\tau,$$

for $y \neq 0$, $j \in \boldsymbol{N}$ and $v \geq 0$. Here $\pm ia_j$, $j \in \boldsymbol{N}$, are simple zeros of the function

$$\varphi(u) = \int_{-\infty}^\infty e^{-2(w^{2m} - uw)} dw.$$

It is known that the function φ has infinitely many zeros ([3]) and all of them exist on the imaginary axis ([8]). Thus we give them the order: $0 < a_j < a_{j+1}$ for $j \in \boldsymbol{N}$. It is easy to check that the S_j^v's are not real analytic on $\{(0 + iy, 0) ; y \in \boldsymbol{R}\}$. Besides this, the S_j^v's, off the set $\{y = 0\}$, belong to sth order Gevrey class G^s for all $s \geq 2m$, but no better, where $G^s := \{f ; \exists C > 0 \text{ s.t. } |\partial^\alpha f| \leq C^{|\alpha|} \Gamma(s|\alpha|)\ \forall \alpha\}$.

Let $S((z, t) ; (w, s))$ be the Szegö kernel of M; that is, the distribution kernel associated to the operator defined by the orthogonal projection of $L^2(\boldsymbol{C} \times \boldsymbol{R})$, with respect to the Lebesgue measure, onto the kernel of $\bar{\partial}_b$. Define the distribution $K(z, t) = S((z, t) ; (0,0))$, then K is C^∞ function away from (0,0) ([6], [7]).

In case $M = M_m (m = 2,3, \ldots)$, we give a representation of K in terms of the singular solutions $\{S_j^v\}_{j \in \boldsymbol{N}}$.

Theorem 2.1. *If* $|\arg z \pm \frac{\pi}{2}| < \frac{1}{2m-1}\frac{\pi}{2}$, *then*

$$K(z, t) = c\int_0^\infty e^{-p} H(z, t ; p) dp, \tag{2.1}$$

where

$$H(z, t ; p) = \sum_{j=1}^{\infty} c_j S_j^{\frac{1}{m}}(z, t) p^{f(j)}, \tag{2.2}$$

for some sequence $f(j) = j + j_0 + O\left(\frac{1}{j}\right)$ (> 0) *as* $j \to \infty$ *and* c, c_j*'s and* $j_0 \in \boldsymbol{Z}$ *are constants. Moreover there exists a positive constant* $C(z)$ *depending on* z *such that,*

$$|H(z, t ; p)| \leq C(z) |p|^{-\frac{1}{4}}.$$

Let $B((z_1, z_2) ; (w_1, w_2))$ be the Bergman kernel of the domain $D = \{\mathrm{Im} z_2 > P(z_1)\} \subset \boldsymbol{C}^2$; that is, the distribution kernel for the orthogonal projection of $L^2(D)$ onto the subspace of holomorphic functions. Then B extends to a C^∞ function on $\bar{D} \times \bar{D}$ minus the diagonal ([6], [7]). When $D = D_m := \{\mathrm{Im} z_2 > [\mathrm{Re} z_1]^{2m}\} (m = 2,3, \ldots)$, we obtain a similar representation for the distribution $K^B(z, t) := B((z, t + ix^{2m}) ; (0, 0))$.

Theorem 2.2. *If* $|\arg z \pm \frac{\pi}{2}| < \frac{1}{2m-1}\frac{\pi}{2}$, *then*

$$K^B(z, t) = c^B \int_0^\infty e^{-p} H^B(z, t ; p) dp, \tag{2.3}$$

with

$$H^B(z, t ; p) = \sum_{j=1}^{\infty} c_j S_j^{1+\frac{1}{m}}(z, t) p^{f(j)}, \tag{2.4}$$

where c_j's *and* $f(j)$ *are as in Theorem* 2.1 *and* c^B *is a constant. Moreover there exists a positive constant* $C^B(z)$ *depending on* z *such that,*

$$|H^B(z, t ; p)| \leq C^B(z) |p|^{-\frac{1}{4}}.$$

Remarks. 1) As a corollary of the above theorems, we can obtain directly that K and K^B are not real analytic on $\{(0 + iy, 0) ; y \in \boldsymbol{R}\}$ and they belong to G^s for all $s \geq 2m$, but no better, away from (0,0).

2) If we change the order of the sum and the integral in (2.1), (2.3) formally, we obtain the formal sum of the form $\sum_{j=1}^\infty d_j S_j^v(z, t)$, where d_j's are constants. However the formal sum is not convergent in the usual sense.

3) The constants c_j's in (2.2), (2.4) are given as follows:

$$c_j = \frac{1}{\varphi'(ia_j)} \frac{1}{\Gamma(f(j) + 1)} \quad j \in \boldsymbol{N}.$$

4) We conjecture that j_0 and $O\left(\frac{1}{j}\right)$ in the above theorems are removable, that is, $f(j)$ can be replaced by j.

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