Monopoles and Dipoles in Biharmonic Pseudo Process

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Δ^2 denotes the differential operator Laplacian square. It is called *the biharmonic operator* and plays an important role in the theory of elasticity and fluid dynamics. Given an equation

$$(1)\quad \partial_t u(t,x) = -\Delta^2 u(t,x) \equiv -\partial_x^4 u(t,x),\quad t>0,\ x\in\mathbf{R}^1,$$

we easily obtain its fundamental solution $p(t,x)$;

$$(2)\quad p(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \exp\{-i\xi x - \xi^4 t\},\quad t>0,\ x\in\mathbf{R}^1.$$

Following the pioneer works of Krylov [2] and Hochberg [1], we consider 'particles' whose 'transition probability density' is taken to be $p(t,x)$ though $p(t,x)$ is not positive. We call these 'particles' *biharmonic pseudo process* (or BPP in short). In this note, we shall calculate a 'distribution' of the first hitting time of BPP, and it will be proved that BPP observed at a fixed point behaves as a mixture of particles of two different types, which are *'monopoles'* and *'dipoles'*.

1. Given positive t and s, $p(t,x)$ is an even function in x belonging to the Schwartz class $\mathcal{S}$, $p(t,x) = t^{-1/4}p(1, x/t^{1/4})$, $\int_{-\infty}^{\infty} dx\, p(t,x) = 1$, and $\int_{-\infty}^{\infty} dy\, p(t, y-x)\, p(s,y) = p(t+s, x)$. Here, note that values of $p(t,x)$ may be negative. In fact, Hochberg [1] proved that

$$p(1,|x|) = a|x|^{-1/3}\exp\{-b|x|^{4/3}\}\times \cos c|x|^{4/3} + \text{a lower order term}$$

for large $|x|$, where a, b, and c are positive constants. From the above, we see that

$$(3)\quad \int_{-\infty}^{\infty}|p(t,x)|dx = \int_{-\infty}^{\infty}|p(1,x)|\,dx = \text{a constant} \equiv \rho > 1.$$

Basing on this $p(t,x)$, we can build up a finitely additive signed measure $\tilde{\boldsymbol{P}}_x$ on cylinder sets in $\mathbf{R}^{[0,\infty)}$. *A cylinder set* in $\mathbf{R}^{[0,\infty)}$, say Γ, is a set such that

$$(4)\quad \Gamma = \{\omega\in\mathbf{R}^{[0,\infty)} : \omega(t_1)\in B_1,\dots,\omega(t_n)\in B_n\}$$

where $0\le t_1<\cdots<t_n$ and B_k's are Borel sets in $\mathbf{R}^1$. For each cylinder set as in (4), we define a finitely additive signed measure $\tilde{\boldsymbol{P}}_x$ by

$$(5)\quad \tilde{\boldsymbol{P}}_x[\Gamma] \equiv \int_{B_1} dy_1\cdots\int_{B_n} dy_n\, p(t_1, y_1-x)\times p(t_2-t_1, y_1-y_2)\cdots p(t_n-t_{n-1}, y_n-y_{n-1}).$$

Fix $0\le t_1<\cdots<t_n$, and $\tilde{\boldsymbol{P}}_x$ is a σ-additive finite measure on a Borel field on $\mathbf{R}^n$ with total variation ρ^n. We say that a function f defined on $\mathbf{R}^{[0,\infty)}$ is *tame*, if

$$(6)\quad f(\omega) = g(\omega(t_1),\dots,\omega(t_n)),\ \omega\in\mathbf{R}^{[0,\infty)}$$

for a Borel function g on $\mathbf{R}^n$ with $0\le t_1<\cdots<t_n$. For each tame function f as in (6), we define its expectation in the ordinary way;

$$(7)\quad \tilde{\boldsymbol{E}}_x[f(\omega)] = \int f(\omega)\,\tilde{\boldsymbol{P}}_x[d\omega(t_1)\times\cdots\times d\omega(t_n)]$$

if the right hand side exists.

$\tilde{\boldsymbol{P}}_x$ satisfies the consistency condition, but (3) disturbs validity of Kolmogorov's extension theorem. So we do not know exactly an existence of a σ-additive extension of (5) into a function space. But we easily see that total variation of such σ-additive extension must be infinite if it may exist.

2. We extend an expectation given by (7). Let n and N be natural numbers. For each $\omega\in\mathbf{R}^{[0,\infty)}$, we set

$$\omega_n^N(t) \equiv \begin{cases} \omega\left(\dfrac{k}{2^n}\right) & \text{if } \dfrac{k}{2n}\le t<\dfrac{k+1}{2^n} \text{ and } t<N \\ \omega(N) & \text{if } t\ge N. \end{cases}$$

This approximating function ω_n^N is a step function in *Skorokhod space* $\mathbf{D}[0,\infty)$, which is a space of all right continuous functions on $[0,\infty)$ with left hand limits.

Definition 1. We say that a function F on $\mathbf{R}^{[0,\infty)}$ is *admissible* if F satisfies the following;

(a) for each n and N, $F(\omega_n^N)$ is tame,

(b) for each $\omega\in\mathbf{R}^{[0,\infty)}$, $\lim_{n\to\infty}\lim_{N\to\infty}F(\omega_n^N) = F(\omega)$,

(c) there exists $\lim_{n\to\infty}\lim_{N\to\infty}\sum_{k=1}^{N}|\tilde{\boldsymbol{E}}_x[F(\omega_n^k)] - \tilde{\boldsymbol{E}}_x[F(\omega_n^{k-1})]|$.

For each admissible function F, we define its expectation by

$$(8)\quad \boldsymbol{E}_x[F(\omega)] \equiv \lim_{n\to\infty}\lim_{N\to\infty}\tilde{\boldsymbol{E}}_x[F(\omega_n^N)].$$

If exists, this expectation is unique owing to (c) of Definition 1. When a defining function $I_A(\omega)$ is admissible for a subset $A \subset \mathbf{R}^{[0,\infty)}$, we denote $\boldsymbol{E}_x[I_A(\omega)] \equiv \boldsymbol{P}_x[A]$.

We denote by $\mathbf{H}^\alpha[0, \infty)$ a set of all functions on $[0, \infty)$ which satisfy Hölder's condition of order α. In [2], Krylov proved that

$$\text{Total variation of } \boldsymbol{P}_x[\mathbf{H}^\alpha[0,\infty)^c] = 0 \text{ if } \alpha < \frac{1}{4}.$$

This tells us that mass of $\boldsymbol{P}_x$ is concentrated in continuous functions. However for a technical reason, we take a larger Skorokhod space $\mathbf{D}[0, \infty)$ as *a path space* of the pseudo process corresponding to $\boldsymbol{P}_x$. From now on, we confuse those $\boldsymbol{P}_x$ and $\boldsymbol{E}_x$ with their respective restrictions on $\mathbf{D}[0, \infty)$.

Definition 2. *A biharmonic pseudo process*, or *BPP* in short, is a family of finitely additive signed measures $\{\boldsymbol{P}_x : x \in \mathbf{R}^1\}$ which is defined on subsets in $\mathbf{D}[0, \infty)$ whose defining functions are admissible.

Remark 3. The domain of $\boldsymbol{P}_x$ is a finitely additive field in $\mathbf{D}[0, \infty)$ which strictly includes all cylinder sets in $\mathbf{D}[0, \infty)$.

3. Given $\omega \in \mathbf{D}[0, \infty)$, we put

$$\tau_0(\omega) \equiv \inf\{t > 0 : \omega(t) < 0\},$$

which is called *the first hitting time to a set* $(-\infty, 0)$. Following to (8), we shall calculate an expectation of a function

$$\exp\{-\lambda\tau_0(w) + i\beta\omega(\tau_0)\} \tag{9}$$

for each $\lambda > 0$ and $\beta \in \mathbf{R}^1$. Although our $\boldsymbol{E}_x$ in (8) is not an integral by a usual probability measure, we can calculate the expectation of (9) by Spitzer's identity [4], which has been proved by a combinatorial method.

Proposition 4. *Let* $x \geq 0$, $\lambda > 0$, *and* $\beta \in \mathbf{R}^1$. *Then a function* (9) *is admissible, and we have*

$$\begin{aligned} &\boldsymbol{E}_x[\exp\{-\lambda\tau_0(\omega) + i\beta\omega(\tau_0)\}] \qquad (10)\\ &= \frac{1}{\sqrt{2}}[\bar\theta_1 \exp\{\lambda^{1/4}\theta_2 x\} + \theta_1 \exp\{\lambda^{1/4}\bar\theta_2 x\}] \\ &+ \frac{i\beta}{\sqrt{2}\lambda^{1/4}}[-i\exp\{\lambda^{1/4}\theta_2 x\} + i\exp\{\lambda^{1/4}\bar\theta_2 x\}], \end{aligned}$$

where $\theta_1 \equiv \exp\{\pi i/4\}$ *and* $\theta_2 \equiv \exp\{3\pi i/4\}$.

Remark 5. Hochberg [1] already computed an expectation of $\exp\{-\lambda\tau_0(\omega)\}$ in two ways, and obtained different results each other. His first method is based on Spitzer's identity, and the other is on André's reflection principle. As a matter of fact, André's reflection principle does not work for *BPP* (see Remark 16). So his second result is not right, but his first result coincides with (10) in case $\beta = 0$.

4. In the case of usual probability theory, we can derive a joint distribution

$$\boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da] \tag{11}$$

from its characteristic function (10), by Bochner's theorem. But the theorem is not valid in the case of *BPP*, and we need another way to obtain (11). Given functions ϕ and φ in the Schwartz class $\mathcal{S}$, we prove that a function.

$$\exp\{-\lambda\tau_0(\omega)\}\phi(\tau_0(\omega))\varphi(\omega(\tau_0))$$

is admissible and its expectation is a continuous bilinear functional on the space $\mathcal{S} \times \mathcal{S}$. Then we obtain a Schwartz's tempered distribution $q(x\,;t, a)$ such that

$$\begin{aligned} &\boldsymbol{E}_x[\exp\{-\lambda\tau_0(\omega)\}\phi(\tau_0(\omega))\varphi(\omega(\tau_0))] \qquad (12)\\ &\quad = \int_0^\infty dt \int da\, e^{-\lambda t} q(x : t, a)\phi(t)\varphi(a) \end{aligned}$$

for any ϕ and φ in $\mathcal{S}$. In the usual probability theory, distributions of real-valued random variables are non-negative continuous linear functionals on the function space $\mathbf{C}$. While (12) suggests us that a 'distribution' in *BPP* should be understood as a continuous linear functional on some function space which includes $\mathcal{S}$ at least.

Definition 6. We call the above Schwartz's tempered distribution $q(x\,;t, a)$ *a density* of the 'distribution' (11), and define

$$\boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da] \equiv q(x\,;t, a)\,dt\,da.$$

Theorem 7. *Let* $x \geq 0$. *Then*

$$\begin{aligned} \boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da] &= [K(t, x)\delta(a) \qquad (13)\\ &\quad - J(t, x)\delta'(a)]dt\,da \end{aligned}$$

where $\delta(a)$ *is Dirac's delta function with its derivative* $\delta'(a)$ *in the sense of Schwartz's distribution and*

$$\begin{aligned} K(t, x) &\equiv \frac{1}{2\pi}\int_0^\infty db \exp\{-b^4 t\}4b^3(\sin bx \\ &\qquad - \cos bx + \exp\{-bx\}), \\ J(t, x) &\equiv \frac{1}{2\pi}\int_0^\infty db \exp\{-b^4 t\}4b^2(\sin bx - \\ &\qquad \cos bx + \exp\{-bx\}). \end{aligned}$$

Now we can extend (13) into a continuous bilinear functional on a wider space, owing to its explicit form:

Corollary 8. (13) *is extended into a continuous bilinear functional on* $\mathbf{B}_b[0, \infty) \times \mathbf{C}^1$, *where* $\mathbf{B}_b[0, \infty)$ *is a space of all bounded Borel functions on* $[0, \infty)$ *and* $\mathbf{C}^1$ *is a space of all continuously differentiable functions on* $\mathbf{R}^1$.

Remark 9. (i) The derivative of Dirac's delta function, $-\delta'(a)$, is called a *dipole* in physics, that is a particle carrying equal magnitude

and opposite sign charges.
(ii) In the case of Brownian motion $\{B(t),\ t \geq 0\}$, it is well known that

$$\boldsymbol{P}_x[\tau_0 \in dt,\ B(\tau_0) \in da] = \frac{x}{\sqrt{2\pi t^3}} \exp\{-x^2/2t\} \times \delta(a)\,dt\,da\ ;\ x > 0,$$

where $\delta(a)\,da$ is δ-measure with a point mass at $\{0\}$. Comparing this with (13), we know that *BPP* behaves as if it is composed from particles of two kinds when we observe it at a fixed point. A particle of the first kind, which we call a *monopole*, carries charge of a single sign similar to a Brownian particle. The second is just the same as a dipole in physics and we also call it a *dipole*. Since total variation of $\boldsymbol{P}_x$ is not bounded, dipole exists in case of *BPP*. While Brownian motion has no dipole since Wiener measure has finite total variation.

Now, by this remark, we come to have an intuitive explanation of Theorem 7.

Definition 10. We define two different 'distributions' in *BPP*:

$$\boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \text{ is monopole}] = K(t, x)\,\delta(a)\,dt\,da$$

$$\boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \text{ is dipole}] = J(t, x)\,(-\delta'(a))\,dt\,da,$$

where the former is a continuous linear functional on $\mathbf{B}_b[0, \infty) \times \mathbf{C}$ and the latter on $\mathbf{B}_b[0, \infty) \times \mathbf{C}^1$.

Corollary 11. *In the sense of continuous linear functionals on* $\mathbf{B}_b[0, \infty) \times \mathbf{C}^1$,

$$\begin{aligned}&\boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da]\\ &= \boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \textit{ is monopole}]\\ &+ \boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \textit{ is dipole}].\end{aligned}$$

Let a monopole start from $\{x\}$. Using Corollary 11, we have

$$\begin{aligned}&\int db\,\delta(b-x)\,[\boldsymbol{P}_b[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \text{ is monopole}]\\ &+ \boldsymbol{P}_b[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \text{ is dipole}]]\\ &= [K(t, x)\delta(a) + J(t, x)\,(-\delta'(a))]\,dt\,da,\\ &= \boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da].\end{aligned}$$

On the other hand , if a dipole starts from $\{x\}$, then we have

$$\begin{aligned}(14)\quad &\int db\,(-\delta'(b-x))\,[\boldsymbol{P}_b[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \text{ is monopole}]\\ &+ \boldsymbol{P}_b[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \text{ is dipole}]]\\ &= \left[\frac{\partial K}{\partial x}(t, x)\,\delta(a) + \frac{\partial J}{\partial x}(t, x)\,(-\delta'(a))\right]dt\,da.\end{aligned}$$

Here (14) tells us that not only a monopole but also a dipole produces both monopoles and dipoles.

Whenever we consider both effect of monopoles and dipoles, *BPP* fulfills a strong Markov property with respect to the first hitting time:

Corollary 12. *Let* $y < 0 < x$. *Then in the sense of continuous linear functionals on* $\mathbf{B}_b[0, \infty)$, *it holds that*

$$\begin{aligned}\boldsymbol{P}_x[\omega(t) \in dy] &= \int_{s=0}^{t}\int [\boldsymbol{P}_x[\tau_0(\omega) \in ds,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \text{ is monopole}]\\ &+ \boldsymbol{P}_x[\tau_0(\omega) \in ds,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \text{ is dipole}]\ \boldsymbol{P}_a[\omega(t-s) \in dy]\\ &= \int_{s=0}^{t}\int \boldsymbol{P}_x[\tau_0(\omega) \in ds,\ \omega(\tau_0) \in da]\ \boldsymbol{P}_a[\omega(t-s) \in dy].\end{aligned}$$

In other words, we have a chain rule: For $y < 0 < x$ and each continuously differentiable function f,

$$\begin{aligned}&\int_{t=0}^{\infty}\int \boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_y) \in db]\,f(b)\\ &= \int_{t=0}^{\infty}\int \boldsymbol{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da]\\ &\times \int_{s=0}^{\infty}\int \boldsymbol{P}_a[\tau_y(\omega) \in ds,\ \omega(\tau_y) \in db]\,f(b).\end{aligned}$$

5. We let monopoles and dipoles to be absorbed when they hit the point $\{0\}$. Using results in §2 and §4, we construct a new 'distribution' $\{\boldsymbol{P}_x^0[\omega(t) \in \cdot\,] : x \geq 0\}$, in the following way:

(a) A monopole starts from $x \geq 0$, and moves in the same law as *BPP* until it its $(-\infty, 0)$.
(b) If *BPP* hits $(-\infty, 0)$, it is absorbed either when $\omega(\tau_0)$ is a monopole or a dipole.

We put

$$U^0(x, \lambda, \beta) \equiv \int_0^t dt \int_0^{\infty} \boldsymbol{P}_x^0[\omega(t) \in db]\ \exp\{-\lambda t + i\beta b\}].$$

From the definition of $\boldsymbol{P}_x^0[\omega(t) \in \cdot\,]$, we have

$$\begin{aligned}(15)\quad U^0(x, \lambda, \beta) &= \int_0^{\infty} dt \int_{-\infty}^{\infty} \exp\{-\lambda t + i\beta b\}\ \boldsymbol{P}_x[\omega(t) \in db]\\ &- \int_{t=0}^{\infty} dt \int_{s=0}^{t}\int_{b=-\infty}^{\infty}\int \exp\{-\lambda t + i\beta b\}\\ &\times (\boldsymbol{P}_x[\tau_0(\omega) \in ds,\ \omega(\tau_0) \in da,\ \omega(\tau_0) \text{ is monopole}]\end{aligned}$$

$$+ \boldsymbol{P}_x[\tau_0(\omega) \in ds, \omega(\tau_0) \in da, w(\tau_0) \text{ is dipole}]) \\ \boldsymbol{P}_a[\omega(t-s) \in db].$$

Now after easy calculation, we obtain an explicit form of

$$(16) \int_0^\infty dt \int_0^\infty \boldsymbol{P}_x^0[\omega(t) \in db] \exp\{-\lambda t\} \phi(t) \varphi(b),$$

for each ϕ and φ in $\mathscr{S}$, and it follows that (16) is a continuous bilinear functional on $\mathscr{S} \times \mathscr{S}$. So, as in §4, we obtain a 'distribution' $\boldsymbol{P}_x^0[\omega(t) \in db]$ as a Schwartz's tempered distribution, which can be extended as follows.

Theorem 13. *For* $x, b \geq 0$,

$$(17) \quad \boldsymbol{P}_x^0[\omega(t) \in db] = p(t, b-x)\, db - \\ - \int_{s=0}^t \int \boldsymbol{P}_x[\tau_0 \in ds, \omega(\tau_0) \in da]\, p(s, b-a)\, db,$$

which is a continuous linear functional on $B_b[0, \infty)$.

Remark 14. (i) By Corollary 12, we see that

$$p(t, b-x) - \int_{s=0}^t \boldsymbol{P}_x[\tau_0 \in ds, \omega(\tau_0) \in da] \\ p(s, b-a) = 0, \text{ if } b < 0.$$

Remember that $\boldsymbol{P}_x[\omega(t) \in db] = p(t, b-x)db$, and we may assume that $b \geq 0$ in the theorem.

(ii) We can consider $\boldsymbol{P}_x^0[\omega(t) \in db]$ as a finitely additive signed measure on cylinder sets in $\mathbf{D}[0, \infty)$ in the same way as in (5).

As we expect from the case of Brownian motion with an absorbing barrier boundary condition, the 'distribution' $\boldsymbol{P}_x^0[\omega(t) \in db]$ corresponds to (1) with Dirichlet boundary condition. In fact for a bounded smooth function f on $[0, \infty)$, we put

$$(18) \quad v(t, x) \equiv \int_0^\infty \boldsymbol{P}_x^0[\omega(t) \in db]\, f(b).$$

Corollary 15. *This* v *satisfies*

$$(19) \quad \partial_t v = -\Delta^2 v, t > 0, x > 0; v(0+, x) = f(x)$$

and Dirichlet boundary condition

$$v(t, 0) = 0 = \partial_x v(t, 0), \; t > 0.$$

Remark 16. (i) Since *BPP* observed at a fixed point is composed from monopoles and dipoles, we need two boundary conditions in order to control them respectively. The boundary condition, $v(t, 0) = 0$, means to absorb monopoles, while $\partial_x v(t, 0) = 0$ does to absorb dipoles.

(ii) Owing to Andre's reflection principle, we easily obtain the transition probability of Brownian motion with an absorbing barrier boundary condition. But, by an effect of dipoles, the principle does not work for *BPP*.

As boundary conditions for *BPP*, we may set one of 'absorbing', 'sticking', and various 'reflecting' barrier conditions on each kind of particles respectively. Thus their combinations derive various boundary conditions to (19). For instance, when we set 'usual' reflecting barrier condition for monopoles and absorbing barrier condition for dipoles, we get a pseudo process corresponding to (19) with Neumann boundary condition:

$$\partial_x v(t, 0) = 0 = \partial_x^2 v(t, 0), \; t > 0.$$

We shall discuss the details of these results elsewhere.

References

[1] Hochberg, K. J.: A signed measure on path space related to Wiener measure. Ann. Prob., **6**, 433–458 (1978).

[2] Krylov, Y. Yu.: Some properties of the distribution corresponding to the equation $\partial u / \partial t = (-1)^{q+1} \partial^{2q} u / \partial^{2q} x$. Soviet Math. Dokl., **1**, 760–763 (1960).

[3] Nishioka, K.: The first hitting time and place to a half-line of the biharmonic pseudo process (to appear in Japanese J. Math.).

[4] Spitzer, F.: A combinatorial lemma and its application to probability theory. Trans. Amer. Math. Soc., **82**, 323–339 (1958).