Discretization of Non-Lipschitz Continuous O.D.E. and Chaos

By Yoichi MAEDA and Masaya YAMAGUTI

Ryukoku University

(Communicated by Kiyosi ITÔ, M. J. A., Feb. 13, 1996)

1. Introduction. In the previous paper [1], we investigated Euler's discretization of the scalar autonomous ordinary differential equation which has only one stable equilibrium point. Under some conditions, it is shown that Euler's finite difference scheme $F_{\Delta t}$ is chaotic for a sufficiently large fixed time step Δt.

On the contrary, in this paper, for a sufficiently small fixed time step Δt, we will find the necessary and sufficient conditions under which $F_{\Delta t}$ is stable in the neighborhood of the equilibrium point, and the sufficient conditions under which $F_{\Delta t}$ is chaotic around the equilibrium point.

2. Definitions and assumptions. For the scalar autonomous O.D.E.

(1) $$\frac{du}{dt} = f(u) \quad u \in \boldsymbol{R}^1,$$

we put following assumptions:

$$\begin{cases} f(u) \text{ is continuous in } \boldsymbol{R}^1 \\ f(u) > 0 \quad (u < 0) \\ f(0) = 0 \\ f(u) < 0 \quad (0 < u). \end{cases}$$

In other words, $u = 0$ is the only stable equilibrium point. Euler's discretization scheme for (1) is as follows: with the fixed time step Δt,

$$\frac{x_{n+1} - x_n}{\Delta t} = f(x_n),$$

$$x_{n+1} = x_n + \Delta t \cdot f(x_n).$$

Now, finite difference scheme $F_{\Delta t}(x)$ is defined as

(2) $F_{\Delta t}(x) = x + \Delta t \cdot f(x)$, (i.e. $x_{n+1} = F_{\Delta t}(x_n)$)

and we will investigate this dynamical system $F_{\Delta t}(x)$.

3. Condition for stable behavior of $F_{\Delta t}$. Generally speaking, Euler's finite difference scheme with sufficiently small Δt gives a good approximation for the solution of differential equation. For example, consider a differential equation

$$\frac{du}{dt} = au(1 - u) \quad (u \geq 0,\ a \text{ is a positive constant}).$$

The orbits of the corresponding dynamical system (2) converge to a stable equilibrium point $u = 1$ with any Δt less than $2/a$. But the next example shows that however small Δt is chosen, the orbits don't always converge to the equilibrium point:

$$\frac{du}{dt} = \begin{cases} \sqrt{-u} & (u < 0) \\ -\sqrt{u} & (u \geq 0). \end{cases}$$

In this case, $F_{\Delta t}(x)$ is super-unstable at $x = 0$ ($F'_{\Delta t}(0) = -\infty$), and it has a super-stable orbit ($\pm \Delta t^2/4$) with period 2.

Theorem 1(Lipschitz case). Assume that (1) holds the following additional condition:

(3) $$\left| \frac{f(u)}{-u} \right| < M_0 \quad (\forall u < 0)$$

(M_0 is a positive constant).

Then, there exists $\Delta T > 0$, such that for any $\Delta t (0 < \Delta t < \Delta T)$, $F_{\Delta t}$ has no periodic orbit except the equilibrium point $x = 0$. And for any initial point x_0, $F^n_{\Delta t}(x_0)$ converges to the equilibrium point.

Proof of Theorem 1. Define subsets D_-, D_+, D_0 and D' of $\boldsymbol{R}^2$ by

$$D_- = \{(x, y) \mid x < y < 0\},$$
$$D_+ = \{(x, y) \mid 0 < y < x\}$$
$$D_0 = \{(x, y) \mid 0 < x, y = 0\},$$
$$D' = \{(x, y) \mid y < 0 < x\}.$$

Set $\Delta T = 1/M_0$. From the condition (3), for any $\Delta t (0 < \forall \Delta t < \Delta T)$ and any $x < 0$,

$$F_{\Delta t}(x) = x + \Delta t \cdot f(x) < x + \Delta T \cdot f(x) < x + \Delta T \cdot (-M_0 x) = x(1 - M_0 \Delta T) = 0.$$

On the other hand, $F_{\Delta t}(x) = x + \Delta t \cdot f(x) > x$, so $x < F_{\Delta t}(x) < 0$.

Hence, $x < 0$ implies $(x, F_{\Delta t}(x)) \in D_-$ for any $\Delta t (0 < \forall \Delta t < \Delta T)$.

Let $x_n = F^n_{\Delta t}(x_0)$ $(n \geq 0)$ be an orbit of $F_{\Delta t}$. There are 4 cases of behavior of x_n as follows:

Case (a) $x_0 < 0$. Then $(x_n, x_{n+1}) \in D_-$ for any $n \geq 0$. Therefore the sequence x_n increases monotonously towards the equilibrium point.

Case (b) $x_0 > 0$, and $(x_n, x_{n+1}) \in D_+$ for any $n \geq 0$. Then the sequence x_n decreases monotonously towards the equilibrium point.

Case (c) There exists $N \geq 0$ such that $(x_N, x_{N+1}) \in D_0$. Then $x_{N+1} = x_{N+2} = \cdots = 0$, so x_n also converges to the equilibrium point.
Case (d) There exists $N \geq 0$ such that $(x_N, x_{N+1}) \in D'$. From the fact $x_{N+1} < 0$, this case is reduced to Case (a).
Consequently, x_n converges to the equilibrium point in any case. Q.E.D.

In the above discussion, if we want to show the stability of $F_{\Delta t}$ only in the neighborhood of the equilibrium point, the condition (3) can be eased to $| f(u)/(-u) | < M_0$ $(\exists K < u < 0)$. But this condition is not available for $u \in \boldsymbol{R}^1$. Consider

$$\frac{du}{dt} = \begin{cases} u^2 & (u < 0) \\ -u^2 & (u \geq 0). \end{cases}$$

In this case, for any Δt,

$$F_{\Delta t}^2\left(\pm \frac{2}{\Delta t}\right) = \pm \frac{2}{\Delta t},$$

that is to say, there exists a periodic orbit with period 2.

From the point symmetry at the origin, the condition (3) can be changed to boundedness of $| f(u)/u |$ in the right neighborhood of the equilibrium point. Therefore if neither right nor left limit of $f(u)/u$ is $-\infty$, $F_{\Delta t}^n(x)$ converges to the equilibrium point in its neighborhood with a sufficiently small Δt. On the other hand, if either right and left limit of $f(u)/u$ is $-\infty$, the equilibrium point is super-unstable ($F_{\Delta t}'(0) = -\infty$) with any Δt.

Corollary 1. (i) $F_{\Delta t}^n(x)$ converges to the equilibrium point in its neighborhood with a sufficiently small Δt.

$$\Leftrightarrow \overline{\lim_{u\to -0}} \frac{f(u)}{-u} < +\infty \text{ or } \overline{\lim_{u\to +0}} \frac{-f(u)}{u} < +\infty.$$

(ii) $F_{\Delta t}^n(x)$ never converges to the equilibrium point with any Δt.

$$\Leftrightarrow \lim_{u\to 0} \frac{f(u)}{u} = -\infty.$$

4. Phenomena around the super-unstable equilibrium point. If the limit of $f(u)/u$ is $-\infty$, the equilibrium point of $F_{\Delta t}$ is super-unstable. Moreover, this condition implies the following.

Theorem 2. If $\lim_{u\to 0} \frac{f(u)}{u} = -\infty$,
then, there exists $\Delta T > 0$, such that $F_{\Delta t}(x)$ has a periodic orbit with period 2 for any $\Delta t (0 < \forall \Delta t < \Delta T)$.

Proof of Theorem 2. Let us set $\Delta T = \sup_{x\neq 0} \frac{-x}{f(x)}$ $(0 < \Delta T \leq +\infty)$. For any $\Delta t (0 < \forall \Delta t < \Delta T)$, there exists $x_0 \neq 0$ such that $\Delta t = -x_0/f(x_0)$. Without loss of generality, $x_0 < 0$. Then $F_{\Delta t}(x_0) = x_0 + \Delta t \cdot f(x_0) = 0$, and $F_{\Delta t}^2(x_0) = 0 > x_0$. Hence $x_0 < F_{\Delta t}^2(x_0)$.

On the other hand, if we show the existence of $x_1 (x_0 < \exists x_1 < 0)$ such that $F_{\Delta t}^2(x_1) < x_1$, there exists an orbit with period 2 by the intermediate value theorem. The proof is the following.

At first we can show the existence of $K_1 < 0$ $(x_0 < K_1)$ such that

$$F_{\Delta t}(x) > -x \quad (K_1 < \forall x < 0).$$

In fact, $F_{\Delta t}(x) + x = 2x + \Delta t \cdot f(x) = x\{2 + \Delta t \cdot f(x)/x\}$ is positive for sufficiently small negative x because $x < 0$ and $f(x)/x \to -\infty$ $(x \to -0)$.

In the same way, as $\lim_{u\to+0} f(u)/u = -\infty$, there exists $K_2 > 0$ such that

$$F_{\Delta t}(x) < -x \quad (0 < \forall x < K_2).$$

Now that $\lim_{x\to-0} F_{\Delta t}(x) = 0$, there exists $x_1 (K_1 < \exists x_1 < 0)$ such that $F_{\Delta t}(x_1) < K_2$. Then $F_{\Delta t}(x_1) > -x_1 (> 0)$ because $K_1 < x_1 < 0$, and besides $F_{\Delta t}(F_{\Delta t}(x_1)) < -F_{\Delta t}(x_1)$ for $0 < F_{\Delta t}(x_1) < K_2$.
In this way, it follows $F_{\Delta t}^2(x_1) < -F_{\Delta t}(x_1) < x_1$, and we can show the existence of $x_1 (> x_2)$ such that $F_{\Delta t}^2(x_1) < x_1$. Q.E.D.

Theorem 2 assures the existence of the periodic orbit with period 2 for a sufficiently small Δt. Moreover if there is a periodic orbit with period 3, $F_{\Delta t}$ is chaotic in the sense of Li-Yorke [2]. In this case, Yamaguti-Maeda already proposed an example [3]. Now we show another example which the order of infinitesimal of $f(u)$ is different between $u \to -0$ and $u \to +0$.

Theorem 3. Suppose that $0 < \alpha < 1$ and the following conditions:

(i) $f(u) = O((-u)^\alpha)$ $(u \to -0)$

(ii) $\lim_{u\to+0} \frac{f(u)}{u^\alpha} = -\infty$.

Then there exists $\Delta T > 0$ such that $F_{\Delta t}(x)$ is chaotic in the sense of Li-Yorke for any $\Delta t (0 < \forall \Delta t < \Delta T)$.

Proof of Theorem 3. To prove chaos in the sense of Li-Yorke, it is enough to show the existence of a, $b = F_{\Delta t}(a)$, $c = F_{\Delta t}(b)$ and $d = F_{\Delta t}(c)$ which satisfy $d \leq a < b < c$. From (i), there exists $K > 0$, $L_1 > L_2 > 0$ such that $L_2(-x)^\alpha \le f(x) \le L_1(-x)^\alpha$ $(-K < \forall x < 0)$. Let $b = -(L_2\alpha \cdot \Delta t)^{\frac{1}{1-\alpha}}$ $(b < 0)$. Before discussing about a, let us prepare 2 numbers, say N and Δ_1:

N is an unique positive solution of $N = L_1 N^\alpha + (L_2\alpha)^{\frac{1}{1-\alpha}}$,

(4) Δ_1 is a positive constant such that $N \cdot \Delta t^{\frac{1}{1-\alpha}} < K$ $(0 < \forall \Delta t < \Delta_1)$.

In the following discussion, assume that $0 < \Delta t < \Delta_1$.

Lemma (a). There exists a $(-N \cdot \Delta t^{\frac{1}{1-\alpha}} \le a < b)$ which satisfies $F_{\Delta t}(a) = b$ for any Δt $(0 < \forall \Delta t < \Delta_1)$.

Proof of Lemma (a). This can be proved by the intermediate value theorem.

$$\begin{aligned} &F_{\Delta t}(-N\cdot\Delta t^{\frac{1}{1-\alpha}}) \\ &= -N\cdot\Delta t^{\frac{1}{1-\alpha}} + \Delta t\cdot f(-N\cdot\Delta t^{\frac{1}{1-\alpha}}) \\ &\le -N\cdot\Delta t^{\frac{1}{1-\alpha}} + \Delta t\cdot L_1(N\cdot\Delta t^{\frac{1}{1-\alpha}})^\alpha \text{ (by (4))} \\ &= (L_1N^\alpha - N)\cdot\Delta t^{\frac{1}{1-\alpha}} \\ &= -(L_2\alpha)^{\frac{1}{1-\alpha}}\cdot\Delta t^{\frac{1}{1-\alpha}} = b. \end{aligned}$$

(5) Hence, $F_{\Delta t}(-N\cdot\Delta t^{\frac{1}{1-\alpha}}) \le b$.

On the other hand, $F_{\Delta t}(b) = b + \Delta t\cdot f(b) > b$, so

(6) $F_{\Delta t}(b) > b$.

From (5), (6) and the continuity of $F_{\Delta t}(x)$, (intermediate value theorem)

$$-N\cdot\Delta t^{\frac{1}{1-\alpha}} \le \exists\, a < b \text{ s.t. } F_{\Delta t}(a) = b.$$

Q.E.D. of Lemma (a)

This is why a exists. Note that

$$-K < -N\cdot\Delta t^{\frac{1}{1-\alpha}} \le a < b < 0.$$

Next, let us estimate c $(c = F_{\Delta t}(b) = b + \Delta t\cdot f(b))$. From $-K < b < 0$,

$$\begin{aligned} &b + \Delta t\cdot L_2(-b)^\alpha \le c \le b + \Delta t\cdot L_1(-b)^\alpha \\ &-(L_2\alpha\cdot\Delta t)^{\frac{1}{1-\alpha}} + \Delta t\cdot L_2(L_2\alpha\cdot\Delta t)^{\frac{\alpha}{1-\alpha}} \le c \\ &\quad\le -(L_2\alpha\cdot\Delta t)^{\frac{1}{1-\alpha}} + \Delta t\cdot L_1(L_2\alpha\cdot\Delta t)^{\frac{\alpha}{1-\alpha}} \\ &L_2^{\frac{1}{1-\alpha}}(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}})\cdot\Delta t^{\frac{1}{1-\alpha}} \le c \\ &\quad\le \{L_1(L_2\alpha)^{\frac{\alpha}{1-\alpha}} - (L_2\alpha)^{\frac{1}{1-\alpha}}\}\cdot\Delta t^{\frac{1}{1-\alpha}} \end{aligned}$$

c is positive because of $\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} > 0$. The following constants C_1, C_2 are independent of Δt.

$$C_1 = L_1(L_2\alpha)^{\frac{\alpha}{1-\alpha}} - (L_2\alpha)^{\frac{1}{1-\alpha}},$$
$$C_2 = L_2^{\frac{1}{1-\alpha}}(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}).$$

Finally, from $d = F_{\Delta t}(c) = c + \Delta t\cdot f(c)$, let us show $a - d > 0$.

(7)
$$\begin{aligned} a - d &\ge -N\cdot\Delta t^{\frac{1}{1-\alpha}} - c - \Delta t\cdot f(c) \\ &\ge -N\cdot\Delta t^{\frac{1}{1-\alpha}} - C_1\cdot\Delta t^{\frac{1}{1-\alpha}} - \Delta t\cdot f(c) \\ &= \Delta t^{\frac{1}{1-\alpha}}\cdot\left\{-(N+C_1) - \frac{f(c)}{\Delta t^{\frac{\alpha}{1-\alpha}}}\right\} \\ &= \Delta t^{\frac{1}{1-\alpha}}\cdot\left\{-(N+C_1) + \left(\frac{c}{\Delta t^{\frac{1}{1-\alpha}}}\right)^\alpha\cdot\left(\frac{-f(c)}{c^\alpha}\right)\right\} \\ &\ge \Delta t^{\frac{1}{1-\alpha}}\cdot\left\{-(N+C_1) + C_2^\alpha\cdot\left(\frac{-f(c)}{c^\alpha}\right)\right\} \end{aligned}$$

Since $c \to +0$ $(\Delta t \to +0)$, (7) is positive for a sufficiently small Δt, thus,

$\exists\, \Delta T > 0$ s.t. $d \le a$ $(0 < \forall \Delta t < \Delta T)$ Q.E.D.

An example of Theorem 3 is the following:

$$f(u) = \begin{cases} (-u)^\alpha & (u < 0) \\ -u^\beta & (u \ge 0) \quad (0 < \beta < \alpha < 1). \end{cases}$$

References

[1] Y. Maeda: Euler's discretization revisited. Proc. Japan Acad., **71A**, 58–61 (1995).

[2] T-Y. Li and J. A. Yorke: Period three implies chaos. Amer. Math., Monthly, **82**, 985–992 (1975).

[3] M. Yamaguti and Y. Maeda: On the discretization of O.D.E. Proceedings of ICIAM 95.