An Anticipatory Itô Formula

By Hui-Hsiung KUO*) and Kenjiro NISHI**)

(Communicated by Kiyosi ITÔ, M. J. A., Feb. 13, 1996)

1. Introduction. Let $B(t)$ be a Brownian motion. The well-known Itô formula states that for any C^2-function F on $\boldsymbol{R}$,

$$F(B(t)) = F(B(0)) + \int_0^t F'(B(s))dB(s) + \frac{1}{2}\int_0^t F''(B(s))ds,$$

where $\int_0^t F'(B(s))dB(s)$ is an Itô integral. Suppose θ is a C^2-function on $\boldsymbol{R}^2$. The purpose of this paper is to find an anticipatory Itô formula for $\theta(B(t), B(1))$. It is anticipatory because of the appearance of $B(1)$. In fact, we will give such a formula for $\theta(X(t), B(1))$ with $X(t)$ being a Wiener integral $X(t) = \int_0^t f(s)dB(s)$ such that $f \in L^\infty([0,1])$.

2. Hitsuda-Skorokhod integrals. Let $\mathcal{S}(\boldsymbol{R})$ denote the real Schwartz space on $\boldsymbol{R}$. The standard Gaussian measure on its dual space $\mathcal{S}'(\boldsymbol{R})$ is denoted by μ. Let (L^2) be the complex Hilbert space of square integrable functions on $(\mathcal{S}'(\boldsymbol{R}), \mu)$. Let $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$ be a Gel'fand triple associated with $(\mathcal{S}'(\boldsymbol{R}), \mu)$ (see [2], [5], or [7]). Let ∂_t denote the white noise differentiation. It is a continuous linear operator from $(\mathcal{S})$ into itself. Its adjoint ∂_t^* is a continuous linear operator from $(\mathcal{S})^*$ into itself.

Let g be a weakly measurable function from [0,1] into $(\mathcal{S})^*$ such that $t \mapsto \partial_t^* g(t)$ is Pettis integrable. The integral $\int_0^1 \partial_t^* g(t)dt$ defines a generalized function in $(\mathcal{S})^*$. If it is a random variable in (L^2), then we call it the *Hitsuda-Skorokhod integral* of g ([3], [8]). For instance, if $g \in L^2([0,1] \times \mathcal{S}'(\boldsymbol{R}))$ is nonanticipating, then $\int_0^1 \partial_t^* g(t)dt$ is a Hitsuda-Skorokhod integral. In fact, for such a function g, its Hitsuda-Skorokhod integral agrees with its Itô integral [4], i.e.,

$$\int_0^1 \partial_t^* g(t)dt = \int_0^1 g(t)dB(t),$$

where $B(t)$ is the Brownian motion $B(t, x) = \langle x, 1_{[0,t)}\rangle$, $t \geq 0$, $x \in \mathcal{S}'(\boldsymbol{R})$. In particular, we have the equality

$$\left\| \int_0^1 \partial_t^* g(t)dt \right\|^2 = \int_0^1 \| g(t) \|^2 dt.$$

where $\| \cdot \|$ denotes the (L^2)-norm. However, this equality may hold even if g is not nonanticipating. For example, this equality holds for

$$g(t) = \begin{cases} B(t) + B(1) - B(1-t), & \text{if } 0 \leq t \leq \frac{1}{2}; \\ B(1-t) + B(1) - B(t), & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

3. An anticipatory Itô formula. Let $B(t)$ be the above Brownian motion. We have the following theorem.

Theorem 1. *Let $f \in L^\infty([0,1])$ and let $X(t) = \int_0^t f(s)dB(s)$, $t \in [0,1]$, be the Wiener integral of f. Suppose $\theta(x, y)$ is a C^2-function on $\boldsymbol{R}^2$ such that*

$$\theta(X(\cdot), B(1)), \frac{\partial^2\theta}{\partial x^2}(X(\cdot), B(1)), \frac{\partial^2\theta}{\partial x \partial y}(X(\cdot), B(1)) \in L^2([0, 1] \times \mathcal{S}'(\boldsymbol{R})).$$

Then for any $0 \leq t \leq 1$, the integral $\int_0^t \partial_s^*\left(f(s)\frac{\partial\theta}{\partial x}(X(s), B(1))\right)ds$ is a Hitsuda-Skorokhod integral and the following equality holds in (L^2) for all $0 \leq t \leq 1$,

$$\theta(X(t), B(1)) = \theta(X(0), B(1)) + \int_0^t \partial_s^*\left(f(s)\frac{\partial\theta}{\partial x}(X(s), B(1))\right)ds + \int_0^t \left(\frac{1}{2}f(s)^2\frac{\partial^2\theta}{\partial x^2}(X(s), B(1)) + f(s)\frac{\partial^2\theta}{\partial x \partial y}(X(s), B(1))\right)ds.$$

To prove this theorem, we first assume that f is a simple function. In this case, we can use

---

*) Department of Mathematics, Louisiana State University, U.S.A.

**) Department of Mathematics, Meijo University.
Research supported by U. S. Army Research Office grant DAAH04-94-G-0249.

the S-transform to verify the above equality. Then we use the approximation method to prove the general case. For details, see the forthcoming book [5].

4. An anticipatory stochastic integral equation. Theorem 2. *Let $a \in \boldsymbol{R}$ and $f \in L^\infty([0, 1])$. Then the stochastic integral equation*

$$X(t) = a + \int_0^t \partial_s^*(f(s)B(1)X(s))ds, \quad 0 \leq t \leq 1,$$

has a unique solution in $L^2([0, 1] \times \mathscr{S}'(\boldsymbol{R}))$ given by

$$X(t) = a \exp\Big[B(1) \int_0^t f(s) e^{-\int_s^t f(\tau)\, d\tau} dB(s) - \frac{1}{2} B(1)^2 \int_0^t f(s)^2 e^{-2\int_s^t f(\tau)\, d\tau} ds - \int_0^t f(s) ds\Big].$$

This stochastic integral equation is anticipatory because $B(1)$ appears in the equation and $t \in [0, 1]$. The existence part of this theorem can be proved by using Theorem 1. For the uniqueness part, we need to use the Wiener-Itô decomposition theorem. For details, see the forthcoming book [5].

Example 1. Consider the stochastic integral equation

$$(1) \quad X(t) = 1 + \int_0^t \partial_s^*(B(1)X(s))ds, \; 0 \leq t \leq 1.$$

By Theorem 2 the solution is given by

$$X(t) = \exp\Big[B(1) \int_0^t e^{-(t-s)} dB(s) - \frac{1}{4} B(1)^2 (1 - e^{-2t}) - t\Big].$$

This solution has been obtained by using a different method in [1].

Example 2. For comparison, consider a similar stochastic integral equation as in Example 1,

$$(2) \quad Y(t) = 1 + \int_0^t \partial_s^*(B(1) \diamond Y(s))ds, \; 0 \leq t \leq 1,$$

where $\diamond$ denotes the Wick product. This equation has a unique solution which can be derived by using the S-transform method,

$$Y(t) = \frac{1}{\sqrt{1 + t + t^2}} \exp\Big[-\frac{1}{2(1 + t + t^2)} (tB(1)^2 - 2(1 + t)B(1)B(t) + B(t)^2)\Big].$$

For the derivation of this solution, see the forthcoming book [5]. We mention that equation (2) has been studied in [6]. But the solution given in [6] is incorrect.

It is interesting to compare the solution $X(t)$ of equation (1) and the solution $Y(t)$ of equation (2). Observe that the randomness of $Y(t)$ comes from only $B(1)$ and $B(t)$, while $X(t)$ depends on $B(1)$ and $B(s)$ for $0 \leq s \leq t$. Moreover, let us examine the effect of $B(1)$ on these solutions for small t by using the following informal expressions

$$B(t) \sim \pm\sqrt{t},$$

$$\int_0^t e^{-(t-s)} dB(s) \sim \pm\frac{1}{\sqrt{2}} (1 - e^{-2t})^{1/2}.$$

Then we can check that

$$\ln X(t) \sim \pm B(1)\sqrt{t} - \frac{1}{2} (2 + B(1)^2)t \pm \frac{1}{2} B(1) t^{3/2} + \frac{1}{2} B(1)^2 t^2 + \cdots,$$

$$\ln Y(t) \sim \pm B(1)\sqrt{t} - \frac{1}{2} (2 + B(1)^2)t + \frac{1}{4} t^2 + \cdots.$$

Thus $B(1)$ has the same effect on $X(t)$ and $Y(t)$ up to order t for small values of t.

5. Concluding remarks. 1. We can easily modify the Itô formula in Theorem 1 to get one for the function $\theta(t, x, y)$ which depends also on t. For the modification, we add one more condition

$$\frac{\partial \theta}{\partial t} (\cdot, X(\cdot), B(1)) \in L^2([0, 1] \times \mathscr{S}'(\boldsymbol{R}))$$

and then add one more term $\int_0^t \frac{\partial \theta}{\partial s} (s, X(s), B(1))ds$ to the right hand side of the formula. In fact, it is this version of Itô's formula that we need to use for the proof of Theorem 2.

2. The Itô formula in Theorem 1 is for the function $\theta(X(t), B(1))$, where $X(t)$ is a Wiener integral. It is important to find out whether such a formula can be established for an Itô integral $X(t)$. Furthermore, suppose θ is a C^2-function satisfying certain conditions. We are interested in an Itô type formula for $\theta(X(t))$, where $X(t)$ is a Hitsuda-Skorokhod integral. Such a formula will be very useful in the white noise distribution theory.

References

[1] R. Buckdahn: Skorokhod's integral and linear stochastic differential equations (1987)(preprint).

[2] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: White Noise. An Infinite Dimensional Calculus. Kluwer Academic Publishers (1993).

[3] M. Hitsuda: Formula for Brownian partial derivatives. Second Japan-USSR Symp. Probab. Th., **2**, 111–114 (1972).
[4] I. Kubo and S. Takenaka: Calculus on Gaussian white noise. III. Proc. Japan Acad., **57A**, 433–437 (1981).
[5] H.-H. Kuo: White Noise Distribution Theory (in preparation).
[6] H.-H. Kuo and J. Potthoff: Anticipating stochastic integrals and stochastic differential equations. White Noise Analysis-Mathematics and Applications (eds. T. Hida, *et al.*). World Scientific, pp. 256–273 (1990).
[7] N. Obata: White Noise Calculus and Fock Space. Lect. Notes in Math., vol. 1577, Springer-Verlag (1994).
[8] A. V. Skorokhod: On a generalization of a stochastic integral. Theory Probab. Appl., **20**, 219–233 (1975).