On Some Examples of Modular QM-abelian Surfaces

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1. Introduction. Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a (normalized) newform of weight two on $\Gamma_0(N)$ with trivial Nebentypus character such that the field of Fourier coefficients $K_f := \boldsymbol{Q}(\{a_n\}_{n=1}^{\infty})$ is a (real) quadratic field. Let A_f denote the associated abelian surface over $\boldsymbol{Q}$ ([12], [13]). Then, $\mathrm{End}_{\boldsymbol{Q}}(A_f) \otimes \boldsymbol{Q}$, the $\boldsymbol{Q}$-algebra of endomorphisms of A_f over $\boldsymbol{Q}$, is exactly K_f. Let $\mathfrak{X}_f$ denote the $\boldsymbol{Q}$-algebra of *all* endomorphisms of A_f: $\mathfrak{X}_f = \mathrm{End}_{\bar{\boldsymbol{Q}}}(A_f) \otimes \boldsymbol{Q}$. If f is a form with complex multiplication, i.e., if there is a Dirichlet character $\psi \neq 1$ such that $a_p = \psi(p) a_p$ for all $p \nmid N$, then $A_f/\bar{\boldsymbol{Q}}$ is the product of two copies of an elliptic curve with complex multiplication by some imaginary quadratic field k, so that $\mathfrak{X}_f = \mathrm{M}_2(k)$. In the following, we always assume that f does not have complex multiplication (and that K_f is a real quadratic field). Then $\mathfrak{X}_f$ is either K_f, $\mathrm{M}_2(\boldsymbol{Q})$, or the quaternion division algebra B_D over $\boldsymbol{Q}$ with discriminant $D > 1$ (see [7], [8]). We say that A_f has *quaternion multiplication* (or simply QM) if $\mathfrak{X}_f = B_D$ for some D.

Definition. Let $f = \sum a_n q^n$ be as above and let χ be a (primitive) Dirichlet character. Then f is said to possess the *extra twist* by χ if the equality

$$a_p^{\sigma} = \chi(p) a_p$$

holds for all $p \nmid N$, where σ is the non-trivial automorphism of $K_f/\boldsymbol{Q}$. In this case, we say that χ is a *twisting character* of f.

Let f be a newform on $\Gamma_0(N)$ satisfying our assumption. Then $f^{\sigma} := \sum a_n^{\sigma} q^n$ is also a newform on $\Gamma_0(N)$. Further, if χ is any primitive quadratic Dirichlet character of conductor r, then $f^{\chi} := \sum a_n \chi(n) q^n$ is a cuspform on $\Gamma_0(N')$, where N' is the least common multiple of N and r^2. See [13] for general background.

Now let f be a newform on $\Gamma_0(N)$ which possesses the extra twist by χ, say. Then χ is quadratic and the square of the conductor of χ divides N, and in fact $f^{\sigma} = f^{\chi}$. It is also easily seen that χ is a unique twisting character of f, since f is a form without complex multiplication.

Proposition 1. *Let f possess the extra twist by χ. Then*

$$\mathfrak{X}_f = \left(\frac{d,\ \chi(-1)r}{\boldsymbol{Q}}\right),$$

where $\left(\frac{a,\ b}{\boldsymbol{Q}}\right)$ is the quaternion algebra over $\boldsymbol{Q}$ with reduced norm form $x^2 - ay^2 - bz^2 + abw^2$, d is the discriminant of K_f and r is the conductor of χ.

Proof. This is a special case of a result of [7], [8]. □

If f does not possess the extra twist, it is known that $\mathfrak{X}_f = K_f$.

Proposition 2. *Let A_f be an abelian surface attached to a newform f of weight two on $\Gamma_0(N)$ and assume that A_f has QM. Let p be a prime divisor of N with $p^{\nu} \| N$. Then*

(1) $2 \leq \nu \leq 10$ *if* $p = 2$,
(2) $2 \leq \nu \leq 5$ *if* $p = 3$,
(3) $\nu = 2$ *if* $p \geq 5$.

Furthermore, N is divisible by 2^5 or by the square of some prime p such that $p \equiv 3 \pmod 4$.

Proof. By assumption, f possesses the extra twist. If N is exactly divisible by a prime, then $\mathfrak{X}_f = \mathrm{M}_2(\boldsymbol{Q})$ by [9], Theorem 2. So $\nu \geq 2$ if A_f has QM. Put

$$s = \left\lceil \frac{\nu}{2} - 1 - \frac{1}{p-1} \right\rceil,$$

where $\lceil x \rceil$ is the least integer $\geq x$. Then by [3], Theorem 5.5, the center of $\mathfrak{X}_f$ contains $\boldsymbol{Q}(\zeta + \zeta^{-1})$ if $p > 2$ (resp. $\boldsymbol{Q}(\zeta^2 + \zeta^{-2})$ if $p = 2$), where ζ is a primitive p^s-th root of unity, hence we obtain the estimate for ν. The last part follows from [9], Theorem 3 and [1], Theorem 7. □

An example of a QM-abelian surface attached to a newform of weight two on $\Gamma_0(N)$ is given by Koike [6]. In this case the level is $243 = 3^5$, $K_f = \boldsymbol{Q}(\sqrt{6})$, $\chi = \left(\frac{-3}{\cdot}\right)$ and $\mathfrak{X}_f = \left(\frac{6,\ -3}{\boldsymbol{Q}}\right) = B_6$. Since there are, as it seems, no other known examples, it might be interesting to find other examples of modular QM-abelian surfaces.

2. Results for $301 \leq N \leq 2000$. Let N be an integer with $301 \leq N \leq 2000$ satisfying the conditions of proposition 2. There are thirty-four such N's. For each of those N's, we have decomposed $S_2^0(N)$, the space of newforms of weight two on $\Gamma_0(N)$, into $\boldsymbol{Q}$-simple factors by means of trace formulas of Hecke operators ([5], [14], [10]). These are summarized inTable I.

Table I. $\boldsymbol{Q}$-simple splitting of $S_2^0(N)$

$N = \Pi p^{\nu}$	splitting of $S_2^0(N)$
$324 = 2^2 3^4$	$(0, 0, 1^3, 1)$
$361 = 19^2$	$(1\cdot3\cdot4, 1\cdot2^4\cdot3)$
$392 = 2^3 7^2$	$(1, 1^2\cdot2, 1\cdot2, 1^2)$
$432 = 2^4 3^3$	$(1, 1^3, 1^2, 1^2)$
$441 = 3^2 7^2$	$(1, 1\cdot2^2, 1\cdot2^2, 1^3)$
$484 = 2^2 11^2$	$(0, 0, 2^3, 1\cdot2)$
$512 = 2^9$	$(2^3, 2^3\cdot4)$
$529 = 23^2$	$(4^2\cdot5, 2^5\cdot3\cdot5)$
$576 = 2^6 3^2$	$(1, 1^3, 1^3, 1^2)$
$648 = 2^3 3^4$	$(1^2, 2^2, 1\cdot2, 1\cdot2)$
$675 = 3^3 5^2$	$(1^2\cdot2, 1^2\cdot2^3, 1^4\cdot2^2, 1\cdot2^2)$
$784 = 2^4 7^2$	$(1^2\cdot2, 1^4\cdot2, 1\cdot2^2, 1^3)$
$800 = 2^5 5^2$	$(1^3, 1^2\cdot2^2, 1^2\cdot2^2, 1^2\cdot2)$
$864 = 2^5 3^3$	$(1^3, 1^3\cdot2, 1^3\cdot2, 1^3)$
$900 = 2^2 3^2 5^2$	$(0, 0, 0, 0, 1^2, 1, 1^2, 1^3)$
$961 = 31^2$	$(2^2\cdot8\cdot16, 2^4\cdot3\cdot4\cdot8\cdot12)$
$968 = 2^3 11^2$	$(1\cdot2^2, 1\cdot2^2\cdot4, 1\cdot2\cdot4, 1^2\cdot2^2)$
$972 = 2^2 3^5$	$(0, 0, 1^2\cdot2\cdot3, 1^2\cdot3)$
$1024 = 2^{10}$	$(2^2\cdot4^2, 2^4\cdot4^2)$
$1089 = 3^2 11^2$	$(2\cdot4, 1^4\cdot2^2\cdot4, 1\cdot2^4\cdot4, 1^6\cdot2^2)$
$1152 = 2^7 3^2$	$(1^4, 1^7, 1^4, 1^5)$
$1225 = 5^2 7^2$	$(2^3\cdot3\cdot4, 1^4\cdot2^4\cdot3,$ $1^2\cdot2^2\cdot3\cdot4^2, 1^4\cdot2^3\cdot3)$
$1296 = 2^4 3^4$	$(1^3\cdot2, 1\cdot2^3, 1^6, 1^2\cdot2)$
$1323 = 3^3 7^2$	$(1\cdot3\cdot4^2, 1^9\cdot2^2\cdot3,$ $1^2\cdot2\cdot3\cdot4^2, 1^7\cdot2\cdot3)$
$1444 = 2^2 19^2$	$(0, 0, 1\cdot2\cdot6\cdot8, 1^2\cdot2^2\cdot6)$
$1521 = 3^2 13^2$	$(1^2\cdot2\cdot6, 2\cdot4^2\cdot6, 1^3\cdot2^3\cdot3^3, 2^3\cdot3^3)$
$1568 = 2^5 7^2$	$(2^4, 1^6\cdot2^3, 2^4\cdot4, 1^3\cdot2^3)$
$1600 = 2^6 5^2$	$(1^8, 1^6\cdot2^2, 1^5\cdot2^2, 1^6\cdot2)$
$1728 = 2^6 3^3$	$(1^5\cdot2, 1^7\cdot2, 1^9, 1^7)$
$1764 = 2^2 3^2 7^2$	$(0, 0, 0, 0, 1\cdot4, 1^2, 1^2\cdot2, 1^6)$
$1800 = 2^3 3^2 5^2$	$(1^2, 1^3, 1^4, 1^3, 1^2, 1^3, 1^3, 1^4)$
$1849 = 43^2$	$(1^2\cdot2^2\cdot3^2\cdot10\cdot18\cdot20,$ $1^2\cdot2^2\cdot3^2\cdot18\cdot20^2)$
$1936 = 2^4 11^2$	$(1^2\cdot2^3\cdot4, 1^3\cdot2^4\cdot4, 1\cdot2^6, 1^6\cdot2^2)$
$1944 = 2^3 3^5$	$(1^2\cdot2^2\cdot3, 1^3\cdot6, 1^2\cdot2^2\cdot6, 1^3\cdot3)$

The second column must be read as in [2]. Here we adopt the multiplicative notation instead of the additive one. There are 154 two-dimensional $\boldsymbol{Q}$-simple subspaces (and thus 154 $\boldsymbol{Q}$-simple abelian surfaces) in Table I, among which there are only ten (essentially six) subspaces such that the corresponding abelian surface has QM, as explained below.

(1) Let $N = 675 = 3^3\cdot5^2$. In this case, dim $S_2^0(675) = 25$, and there are 4 newforms in $S_2^0(675)$ such that the field of Fourier coefficients is $\boldsymbol{Q}(\sqrt{2})$. Let $f = \sum a_n q^n$ be one of these. Other three forms are obtained by twisting f by $\chi_3 = \left(\frac{-3}{\cdot}\right)$, $\chi_5 = \left(\frac{5}{\cdot}\right)$, $\chi_{15} = \left(\frac{-15}{\cdot}\right)$, respectively. Further, $f^{\sigma} = f^{(3)}$, $(f^{(5)})^{\sigma} = f^{(15)}$, where σ is the non-trivial automorphism of $K_f/\boldsymbol{Q}$, $f^{\sigma} = \sum a_n^{\sigma} q^n$ and $f^{(r)} = \sum a_n \chi_r(n) q^n$. Hence f and $g = f^{(5)}$ possess the extra twist by χ_3. Since $\mathfrak{X}_f = \mathfrak{X}_g = \left(\frac{2, -3}{\boldsymbol{Q}}\right) = B_6$, A_f and A_g are QM-abelian surfaces. Note that A_g is "essentially" the same with A_f in the sence that A_g is obtained by twisting A_f by χ_5. We list below Fourier coefficients a_p of f for $p \leq 173$ (Table II), and characteristic polynomials $\Phi_{T(p)}(X)$ of Hecke operators $T(p)$ on each $\boldsymbol{Q}$-simple subspace of $S_2^0(675)$ for prime $p \leq 19$ (Table III). In that table, signatures $(+, +)$ etc. indicate the signatures of eigenvalues of Atkin-Lehner's involutions W_{27} and W_{25} ([1]).

Table II. Fourier coefficients of $f = \sum a_n q^n$

p	2	3	5	7	11	13	17	19	23	29
a_p	$\sqrt{2}$	0	0	3	$3\sqrt{2}$	3	$-2\sqrt{2}$	1	$5\sqrt{2}$	$-3\sqrt{2}$
p	31	37	41	43	47	53	59	61	67	71
a_p	2	9	$-3\sqrt{2}$	6	$-2\sqrt{2}$	$-7\sqrt{2}$	$-6\sqrt{2}$	-13	-3	$-9\sqrt{2}$
p	73	79	83	89	97	101	103	107	109	113
a_p	9	-5	$-\sqrt{2}$	0	3	$-6\sqrt{2}$	3	$-5\sqrt{2}$	-8	$-\sqrt{2}$
p	127	131	137	139	149	151	157	163	167	173
a_p	-18	$6\sqrt{2}$	$\sqrt{2}$	13	$12\sqrt{2}$	-1	6	-9	$4\sqrt{2}$	$-10\sqrt{2}$

Table III. Characteristic polynomials $\Phi_{T(p)}(X)$ of $T(p) \mid S_2^0(675)$

	(+, +)			(−, −)		
$\Phi_{T(2)}(X)$	X	$X+1$	X^2+X-3	X^2-2	$X-1$	X^2+3X+1
$\Phi_{T(7)}(X)$	$X-1$	X	$X^2+2X-12$	$(X+3)^2$	X	X^2
$\Phi_{T(11)}(X)$	X	$X+5$	$X^2+2X-12$	X^2-18	$X+5$	X^2
$\Phi_{T(13)}(X)$	$X+5$	$X-5$	X^2+6X-4	$(X+3)^2$	$X+5$	X^2
$\Phi_{T(17)}(X)$	X	$X+4$	X^2+4X-9	X^2-8	$X-4$	$X^2+12X+31$
$\Phi_{T(19)}(X)$	$X+7$	$X+2$	X^2-13	$(X-1)^2$	$X+2$	$X^2+4X-41$

	(+, −)				
$\Phi_{T(2)}(X)$	X	X^2-2	X^2-7	$X+1$	X^2-3X+1
$\Phi_{T(7)}(X)$	$X+4$	$(X-3)^2$	$(X-3)^2$	X	X^2
$\Phi_{T(11)}(X)$	X	X^2-18	X^2-28	$X-5$	X^2
$\Phi_{T(13)}(X)$	$X-5$	$(X-3)^2$	$(X+2)^2$	$X+5$	X^2
$\Phi_{T(17)}(X)$	X	X^2-8	X^2-28	$X+4$	$X^2-12X+31$
$\Phi_{T(19)}(X)$	$X-8$	$(X-1)^2$	$(X-1)^2$	$X+2$	$X^2+4X-41$

	(−, +)					
$\Phi_{T(2)}(X)$	X	X^2-7	$X+2$	$X-2$	$X-1$	X^2-X-3
$\Phi_{T(7)}(X)$	$X-4$	$(X+3)^2$	$X-3$	$X-3$	X	$X^2+2X-12$
$\Phi_{T(11)}(X)$	X	X^2-28	$X-2$	$X+2$	$X-5$	$X^2-2X-12$
$\Phi_{T(13)}(X)$	$X+5$	$(X-2)^2$	$X-5$	$X-5$	$X-5$	X^2+6X-4
$\Phi_{T(17)}(X)$	X	X^2-28	$X+8$	$X-8$	$X-4$	X^2-4X-9
$\Phi_{T(19)}(X)$	$X-8$	$(X-1)^2$	$X-1$	$X-1$	$X+2$	X^2-13

Table IV. Fourier coefficients of $f = \sum a_n q^n$

p	2	3	5	7	11	13	17	19	23	29
a_p	0	0	$3\sqrt{2}$	2	$3\sqrt{2}$	-1	$-3\sqrt{2}$	5	$-3\sqrt{2}$	$-6\sqrt{2}$
p	31	37	41	43	47	53	59	61	67	71
a_p	-7	-4	$-6\sqrt{2}$	5	0	$3\sqrt{2}$	$-3\sqrt{2}$	5	11	$-3\sqrt{2}$
p	73	79	83	89	97	101	103	107	109	113
a_p	-1	11	$9\sqrt{2}$	$-12\sqrt{2}$	-7	$6\sqrt{2}$	11	$-6\sqrt{2}$	-1	0
p	127	131	137	139	149	151	157	163	167	173
a_p	-1	$3\sqrt{2}$	$12\sqrt{2}$	-10	$-3\sqrt{2}$	-13	-7	14	$-18\sqrt{2}$	0

(2) Let $N = 972 = 2^2 \cdot 3^5$. In this case, dim $S_2^0(972) = 12$ and there is exactly one two-dimensional $\boldsymbol{Q}$-simple factor. Let $f = \sum a_n q^n$ be (one of) the corresponding newforms. Then we can see that $K_f = \boldsymbol{Q}(\sqrt{2})$ and f possesses the extra twist by $\left(\frac{-3}{\cdot}\right)$; that is, A_f is a QM-abelian surface with $\mathfrak{X}_f = \left(\frac{2, -3}{\boldsymbol{Q}}\right) = B_6$, as in the case $N = 675$. We only include a table of Fourier coefficients of f for $p \leq 173$ (Table IV above) and we will omit that of characteristic polynomials.

(3) There are other examples of modular QM-abelian surfaces such that $\mathfrak{X}_f = B_6$ for $N =$ 1323,1568 (two factors in each) and $N = 1849$ (one factor). First few a_p's for the corresponding f are given in Table V, in which χ is the twisting character of $f = \sum a_n q^n$, and the "sign" is the signature of eigenvalues of Atkin-Lehner's involutions. Note that for $N = 1323$ and $N =$ 1568, one of A_f is obtained by twisting the other by $\left(\frac{-7}{\cdot}\right)$.

(4) We have also found examples of modular QM-abelian surfaces such that $\mathfrak{X}_f \neq B_6$. More precisely, there are two factors such that $\mathfrak{X}_f = B_{14}$ in $S_2^0(1568)$. First few a_p's for the corresponding f are given in Table VI; χ and the "sign" are as in (3). Also, in this case, one of A_f is obtained by twisting the other by $\left(\frac{-7}{\cdot}\right)$.

Table V. Fourier coefficients of $f = \sum a_n q^n$

$N = \Pi p^\nu$	χ	sign	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}
$1323 = 3^3 7^2$	$\left(\frac{-3}{\cdot}\right)$	$(+, -)$	$\sqrt{6}$	0	$-\sqrt{6}$	0	$2\sqrt{6}$	4	$\sqrt{6}$
		$(-, +)$	$\sqrt{6}$	0	$\sqrt{6}$	0	$2\sqrt{6}$	-4	$-\sqrt{6}$
$1568 = 2^5 7^2$	$\left(\frac{-4}{\cdot}\right)$	$(-, +)$	0	$\sqrt{3}$	1	0	$3\sqrt{3}$	0	5
		$(-, -)$	0	$-\sqrt{3}$	-1	0	$3\sqrt{3}$	0	-5
$1849 = 43^2$	$\left(\frac{-43}{\cdot}\right)$	$(+)$	$\sqrt{6}$	$-\sqrt{6}$	$-\sqrt{6}$	$\sqrt{6}$	-1	-3	-7

Table VI. Fourier coefficients of $f = \sum a_n q^n$

$N = \Pi p^\nu$	χ	sign	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}
$1568 = 2^5 7^2$	$\left(\frac{-4}{\cdot}\right)$	$(+, +)$	0	$\sqrt{7}$	-3	0	$-\sqrt{7}$	-4	1
		$(+, -)$	0	$-\sqrt{7}$	3	0	$-\sqrt{7}$	4	-1

3. Additional results. We have the complete list of modular QM-abelian surfaces over $\boldsymbol{Q}$ for $N \leq 3000$. There are 8 two-dimensional $\boldsymbol{Q}$-simple subspaces in the range $2001 \leq N \leq 3000$ such that the corresponding abelian surface has QM; namely, four cases in $N = 2592$, and two cases in $N = 2601$ and in $N = 2700$. They have QM by B_6 except when $N = 2700$. It is, however, worth mentioning that there appear various combinations of $(d, \chi(-1)r)$ such that $\left(\frac{d, \chi(-1)r}{\boldsymbol{Q}}\right) = B_6$ (see section 1 for notation). Here we only include a table of first few Fourier coefficients for $N = 2700$ (Table VII); in this case, $\mathfrak{X}_f = \left(\frac{10, -3}{\boldsymbol{Q}}\right) = B_{10}$, and one of A_f is obtained by twisting the other by $\left(\frac{5}{\cdot}\right)$.

Table VII. Fourier coefficients of $f = \sum a_n q^n$

$N = \Pi p^\nu$	χ	sign	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}
$2700 = 2^2 3^3 5^2$	$\left(\frac{-3}{\cdot}\right)$	$(-, +, -)$	0	0	0	-1	$\sqrt{10}$	-3	$-2\sqrt{10}$
		$(-, -, -)$	0	0	0	1	$\sqrt{10}$	3	$2\sqrt{10}$

4. Remark. Let f be a newform such that its Nebentypus character is *non-trivial* and real quadratic. Then the corresponding abelian variety A_f is isogenous over $\bar{\boldsymbol{Q}}$ to $B \times B$ for some abelian variety B. Especially, if K_f is a CM-field of degree four, B is two-dimensional, and there is an example such that B is a QM-abelian surface (not defined over $\boldsymbol{Q}$), see [11].

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