Zeta Functions of Categories

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(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1996)

Selberg introduced zetas of Riemannian spaces in analogue to arithmetic zetas. His zetas are particularly interesting in view of the explicit trace formulas leading to the determinant expression of zetas. In this paper we look at zetas of categories generalizing both of arithmetic zetas and Selberg zetas, and we try to unify previous zetas. Hopefully categorical view point will supply a new way to investigate fundamental nature of zetas: construction, anlytic continuation, functional equation, special values, zeros and poles. Our study was reported in "Riemann Hypothesis Symposium" (Seattle, August 12–15, 1996).

§1. Categorical zeta functions. We define a zeta function of a category with zeros. We recall basic notions about categories following MacLane [4]. Fix a category C. For objects X and Y, we denote by $\mathrm{Hom}(X, Y)$ the set of morphisms from X to Y. An object X is called initial (resp. terminal) if $\mathrm{Hom}(X, Y)$ (resp. $\mathrm{Hom}(Y, X)$) consists of only one element for every object Y. We say that an object X is a zero object if X is initial and terminal. It follows that all the zero objects of C are isomorphic.

From now on we let C be a category with a zero object. We say that an object X is simple when $\mathrm{Hom}(X, Y)$ is consisting of monomorphisms and zero-morphisms for every object Y. The norm $N(X)$ of an object X is defined as

$$N(X) = \#\mathrm{End}(X) = \#\mathrm{Hom}(X, X)$$

where $\#\ \mathrm{End}(X)$ is the cardinality of endomorphisms of X. We say that a non-zero object X is finite if $N(X)$ is finite. We denote by $P(C)$ the isomorphism classes of all finite simple objects of C. Remark that for each $P = [X] \in P(C)$ the norm $N(P) = N(X)$ is well-defined.

We define the zeta function $\zeta(s, C)$ of C as

$$\zeta(s, C) = \prod_{P \in P(C)} (1 - N(P)^{-s})^{-1}.$$

Theorem 1. This zeta function $\zeta(s, C)$ is categorically defined: if C_1 and C_2 are equivalent categories then $\zeta(s, C_1) = \zeta(s, C_2)$.

The fundamental example is the case of the category of all abelian groups ($\boldsymbol{Z}$-modules) $C = \boldsymbol{Ab} = \mathrm{Mod}(\boldsymbol{Z})$. In this case $P(\boldsymbol{Ab}) = \{\boldsymbol{Z}/p\boldsymbol{Z} \mid p$ runs over the prime numbers$\}$ and $N(\boldsymbol{Z}/p\boldsymbol{Z}) = p$. Thus we rediscover the Riemann zeta function $\zeta(s, \boldsymbol{Ab}) = \zeta(s)$. This categorical interpretation was indicated in [1] [2] [3].

§2. Some examples. Theorem 2. Let R be a finitely generated commutative ring, and let $\mathrm{Mod}(M_n(R))$ be the category of all left $M_n(R)$-modules, where $M_n(R)$ denotes the matrix ring of size n. Then

$$\zeta(s, \mathrm{Mod}(M_n(R))) = \prod_{\boldsymbol{m}} (1 - N(\boldsymbol{m})^{-s})^{-1}$$

is the usual Hasse zeta function $\zeta(s, R)$, where $\boldsymbol{m}$ runs over all maximal ideals of R and $N(\boldsymbol{m}) = \#(R/\boldsymbol{m})$.

Remark 1. Since $\mathrm{Mod}(M_n(R))$ and $\mathrm{Mod}(R)$ are categorically equivalent ($M_n(R)$ and R are Morita equivalent), their zeta functions are identical. The above situation can be extended to the case of Hasse zeta functions of (not necessarily commutative) schemes also. We can formulate categorically the Lichtenbaum-Beilinson conjecture.

Theorem 3. Let $\boldsymbol{Grp}$ be the category of all groups. Then:

(1) $\zeta(s, \boldsymbol{Grp}) = \prod_{G} (1 - N(G)^{-s})^{-1}$

where G runs over isomorphism classes of finite simple groups and $N(G) = \#\mathrm{End}(G)$.

(2) $\zeta(s, \boldsymbol{Grp})$ is meromorphic in $\mathrm{Re}(s) > 1/3$, and holomorphic there except for the simple pole at $s = 1$.

Remark 2. We use the classification of finite simple groups and the following estimation for all (not necessarily finite) simple groups: $N(G) \geqq \# G$.

§3. Selberg zetas and Hasse zetas of categories. Our zeta function $\zeta(s, C)$ of a category C is obtained by looking at simple objects, and this should be called the Hasse ("O-morphism") zeta function also. We may denote it by $\zeta^{HO}(s, C)$.

Now from the view point of the unification of Hasse zetas and Selberg zetas it is interesting to construct the Selberg ("I-morphism") zeta function $\zeta^{SI}(s, C)$ of C looking at simple "paths". This is possible using the category $\tilde{C}$ of sequences of morphisms

$$\cdots \longrightarrow X_{-2} \xrightarrow{f_{-2}} X_{-1} \xrightarrow{f_{-1}} X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots.$$

We define the Selberg zeta function by

$$\zeta^{SI}(s, C) = \prod_{P \in \tilde{P}(\tilde{C})} (1 - N(P)^{-s})^{-1}$$

where $\tilde{P}(\tilde{C})$ denotes the equivalence classes of finite simple objects of $\tilde{C}$ up to isomorphisms and identification under the translation of indices, and $N(P) = \#\operatorname{End}(P)$. (It is natural to consider translation identifying equivalent "paths".) Then the following Selberg-Hasse unification holds:

Theorem 4. $\zeta^{SI}(s, C) = \zeta^{HO}(s, C)$.

This result will supply a suggestion to unify various zeta functions. In particular we may expect a determinant expression

$$\zeta^{SI}(s, C) = \det(D(C) - (s - (\dim(C)/2)))^{(-1)^{\dim(C)+1}}$$

using a suitable Dirac operator $D(C)$ acting on $L^2(C)$ via $\pi_1(C)$. Such a determinant expression would lead us to the proof of the Riemann hypothesis and the Langlands conjecture.

References

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