Gamelin Constants of Two-sheeted Discs

By Masaru HARA

Department of Mathematics, Meijo University

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For any $0 < \delta < 1$ and n, an n-tuple $\{f_j\}$ of functions $f_1, \ldots, f_n$ in the family $H^\infty(R)$ of bounded holomorphic functions on a Riemann surface R is referred to as a corona datum of index (n, δ) if the following condition is satisfied:

(1) $$\delta \leq (\textstyle\sum_j |f_j|^2)^{1/2} \leq 1.$$

An n-tuple $\{g_j\}$ of functions $g_1, \ldots, g_n$ in $H^\infty(R)$ is said to be a corona solution of the datum $\{f_j\}$ if $\sum_j f_j g_j = 1$. The quantity $C(R\,;n, \delta)$ given by

(2) $$C(R\,;n,\delta) = \sup_{\{f_j\}}(\inf_{\{g_j\}}(\sup_{p\in R}(\textstyle\sum_j |g_j(p)|^2)^{1/2}))$$

will be referred to as the Gamelin constant of R of index (n, δ) where the first supremum is taken with respect to corona data $\{f_j\}$ of index (n, δ) on R and the infimum is taken with respect to corona solutions $\{g_j\}$ of each fixed datum $\{f_j\}$ under the usual convention that $\inf_{\{g_j\}} = \infty$ if there exist no corona solutions $\{g_j\}$ of the datum $\{f_j\}$.

We assume that R is a two-sheeted unlimited covering surface over the unit disc D, which we call a two-sheeted disc. We will show the following

Theorem 1. *For each $0 < \delta < 1$, there exists a constant $C(\delta)$ depending only on δ such that*

(3) $$C(\delta) = \sup_n(\sup_R C(R\,;n,\delta)) < \infty,$$

where n runs over all positive integers and R runs over all two-sheeted discs.

Corollary. *Let R be any two-sheeted disc. Let $\{f_j\}$ be a sequence of functions in $H^\infty(R)$ such that $0 < \delta \leq (\sum_j |f_j|^2)^{1/2} \leq 1$. Then there exists a sequence of functions $\{g_j\}$ in $H^\infty(R)$ and a constant $c(\delta)$ depending only on δ such that $\sum_j f_j g_j = 1$ and $(\sum_j |g_j|^2)^{1/2} \leq c(\delta)$.*

Let (R, π, D) be any two-sheeted disc with projection π. For any f in $H^\infty(D)$, the function $f\cdot\pi$ belongs to $H^\infty(R)$. We identify f with $f\cdot\pi$, so that $H^\infty(D)$ is a subset of $H^\infty(R)$. If R has too many branch points, it holds that $H^\infty(R) = H^\infty(D)$, where Corollary was proved by M. Rosenblum [5] and V. A. Tolokonnikov [6] (cf. [4]).

1. In order to prove Theorem 1, by a normal families argument it is enough to show the following

Theorem 2. *Let R be a two-sheeted disc defined by a two-valued function $\zeta = \sqrt{B}$, where B is a finite Blaschke product whose zeros are all simple. If an n-tuple of*

(4) $$f_j = a_j + b_j\sqrt{B} \quad (j = 1, \ldots, n)$$

is a corona datum of index (n, δ) on R such that a_j and b_j are holomorphic on some neighbourhood of $\bar{D}$, then there exists a corona solution $\{g_j\}$ of $\{f_j\}$ such that

$$(\textstyle\sum_j |g_j|^2)^{1/2} \leq C\delta^{-12},$$

where C is a constant independent of δ, B and n.

We will prove Theorem 2 in §§.2-7. In §.2 we introduce a function ρ, which plays an important role in our proof. In §§.3 and 4 corona solutions are given. By duality, those estimates are reduced to ones of four functions, which are accomplished in §§.5 and 6. Our proof is concluded in §.7.

2. Let $(\cdot,\cdot)$ and $\|\cdot\|$ be the inner product and norm of C^n. Let $a = (a_1, \cdots, a_n)$, $b = (b_1, \cdots, b_n)$ and $f = (f_1, \cdots, f_n)$,

(5) $$\rho = \|a\|^4 + \|b\|^4|B|^2 - (a,b)^2\bar{B} - (b,a)^2B + (\|a\|^2\|b\|^2 - |(a,b)|^2)(|B|^2+1),$$

(6) $$x_j = (\|a\|^2 + \|b\|^2)a_j - \{(a,b) + (b,a)B\}b_j$$

and

(7) $$y_j = -\{(a,b) + (b,a)B\}a_j + (\|a\|^2 + \|b\|^2)Bb_j.$$

Proposition 1. *ρ, x_j and y_j are smooth on some neighbourhood of $\bar{D}$ such that $\rho \geq \delta^4$ and $\sum_j(a_j + b_j\sqrt{B})(\bar{x}_j + \bar{y}_j\sqrt{\bar{B}}) = \rho$.*

Proof. By (1) and (4), we have $\sum_j |a_j + b_j\sqrt{B}|^2 \geq \delta^2$ and $\sum_j |a_j - b_j\sqrt{B}|^2 \geq \delta^2$. Since $2|B| \leq |B|^2 + 1$ and

$$(\textstyle\sum_j |a_j + b_j\sqrt{B}|^2)(\sum_j |a_j - b_j\sqrt{B}|^2)$$
$$= \|a\|^4 + \|b\|^4|B|^2 - (a,b)^2\bar{B} - (b,a)^2B + 2(\|a\|^2 + \|b\|^2 - |(a,b)|^2)|B|,$$

we obtain $\rho \geq \delta^4$. □

We may assume that functions x_j and y_j are smooth and have compact supports in the complex plane $\boldsymbol{C}$.

3. Denote by t the transpose operator of a matrix. For $1 \leq j,\ k \leq n$, set

(8) $h_j = \rho^{-1}(\bar{x}_j + \bar{y}_j\sqrt{B})$ and $h = (h_1, \cdots, h_n)$,

(9) $u_{jk} = \rho^{-2}\{(\bar{x}_j\overline{\partial x_k} - \bar{x}_k\overline{\partial x_j}) + (\bar{y}_j\overline{\partial y_k} - \bar{y}_k\overline{\partial y_j})B\}$ and $u = [u_{jk}]$,

(10) $v_{jk} = \rho^{-2}\{(\bar{x}_j\overline{\partial y_k} - \bar{x}_k\overline{\partial y_j}) + (\bar{y}_j\overline{\partial x_k} - \bar{y}_k\overline{\partial x_j})\}$ and $v = [v_{jk}]$,

(11) $u_{0jk}(z) = \dfrac{1}{\pi}\displaystyle\int\int_C \frac{u_{jk}(\zeta)}{z-\zeta}\, d\xi d\eta$ and $u_0 = [u_{0jk}]$

and

(12) $v_{0jk}(z) = \dfrac{1}{\pi}\displaystyle\int\int_C \frac{v_{jk}(\zeta)}{z-\zeta}\, d\xi d\eta$ and $v_0 = [v_{0jk}]$,

then we have $(\alpha)\ u = -{}^t u$ and $v = -{}^t v$, $(\beta)\ \bar{\partial} u_0 = u$ and $\bar{\partial} v_0 = v$, and $(\gamma)\ u + v\sqrt{B} = ({}^t h)\bar{\partial} h - (\bar{\partial}({}^t h))h$, where $\partial = \partial/\partial z = 2^{-1}(\partial/\partial x - i\partial/\partial y)$ and $\bar{\partial} = \partial/\partial\bar{z} = 2^{-1}(\partial/\partial x + i\partial/\partial y)$. Denote by $A_n(D)$ a set of all n-dimensional square matrices $W = [w_{jk}]$ such that w_{jk} are continuous on $\bar{D}$ and holomorphic on D. Then we have $\bar{\partial}(W + u_0) = u$ and $\bar{\partial}(W + v_0) = v$ for $W \in A_n(D)$.

4. For a matrix-valued function $W = [w_{jk}]$ on a set S, let

$$\| W \|_{\infty,S} = \text{ess.sup}_S(\textstyle\sum_{jk} | w_{jk} |^2)^{1/2}.$$

And, for a vector-valued function $g = (g_1, \cdots, g_n)$ on S, let

$$\| g \|_{\infty,S} = \text{ess.sup}_S(| g_1 |^2 + \cdots + | g_n |^2)^{1/2}.$$

A matrix W is said to be anti-symmetric if $W = -{}^t W$. We will give corona solutions $\{g_j\}$ of the corona datum $\{f_j\}$.

Proposition 2. *Let W_u and W_v be anti-symmetric in $A_n(D)$. Let $\Omega = (W_u + u_0) + (W_v + v_0)\sqrt{B}$ and ${}^t g = {}^t(g_1, \cdots, g_n) = {}^t h + \Omega\, {}^t f$. Then each g_j is continuous on $\bar{R}$ and holomorphic on R such that $\sum_j f_j g_j = 1$ and*

$$\| g \|_{\infty,\partial R} \leq \| h \|_{\infty,\partial R} + \| W_u + u_0 \|_{\infty,\partial D} + \| W_v + v_0 \|_{\infty,\partial D}.$$

Proof. By Proposition 1, $h\,{}^t f = f\,{}^t h = 1$. Since $f\Omega\,{}^t f$ is a one-dimensional and anti-symmetric matrix, $f\Omega\,{}^t f = 0$ and hence $\sum_j f_j g_j = f\,{}^t g = f\,{}^t h + f\Omega\,{}^t f = 1$. The function g is continuous on $\bar{R}$. Except for branch points,

$$\begin{aligned}\bar{\partial}({}^t g) &= \bar{\partial}({}^t h) + \{\bar{\partial}(W_u + u_0) + \bar{\partial}(W_v + v_0)\sqrt{B}\}\,{}^t f\\ &= \bar{\partial}({}^t h) + (u + v\sqrt{B})\,{}^t f\\ &= \bar{\partial}({}^t h) + \{{}^t h\bar{\partial} h - (\bar{\partial}({}^t h))h\}\,{}^t f\\ &= \bar{\partial}({}^t h) + {}^t h(\bar{\partial}(h\,{}^t f)) - (\bar{\partial}({}^t h))(h\,{}^t f) = 0.\end{aligned}$$

Since isolated singular points are removable for bounded holomorphic functions, g is holomorphic on R. □

In order to estimate $\| W_u + u_0 \|_{\infty,\partial D}$ and $\| W_v + v_0 \|_{\infty,\partial D}$, we make use of the following lemmas.

Lemma 1. ([4: p. 290]). *For $w_0 = u_0$ (or v_0) and $w = u$ (or v),*

$$\begin{aligned}\inf\{\| W + w_0 \|_{\infty,\partial D}\,;\ W \in A_n(D) \text{ and } W = -{}^t W\}\\ \leq 2 \sup_\varphi \left(\int\int_D | \varphi |^2 \| w \|^2 \log\frac{1}{| z |}\, dxdy\right)^{1/2}\\ + \sup_\varphi \int\int_D | \varphi |^2 \| \partial w \| \log\frac{1}{| z |}\, dxdy,\end{aligned}$$

where φ runs over all of Hardy class H^2 with norm $\| \varphi \|_2 \leq 1$.

Lemma 2. ([4: p. 290]). *If $w \in C^2(\bar{D})$ such that $w \geq 0$ and $\Delta w \geq 0$ and if $\varphi \in H^2$, then*

$$\left(\int\int_D | \varphi |^2 (\Delta w) \log\frac{1}{| z |}\, dxdy\right)^{1/2} \leq ((2\pi e)\sup_D w)^{1/2} \| \varphi \|_2.$$

5. The following lemmas are elementary.

Lemma 3. $\| a \| \leq 1$ *and* $\| b \| \leq 1$.

Proof. By (1) and (4),

$$2\{| a_j |^2 + | b_j\sqrt{B} |^2\} = | a_j + b_j\sqrt{B} |^2 + | a_j - b_j\sqrt{B} |^2 \leq 2.$$

Hence $\sum_j | a_j |^2 \leq 1$ and $\sum_j | b_j\sqrt{B} |^2 \leq 1$. The function $\sum_j | b_j |^2$ is subharmonic. By the maximum principle,

$$\textstyle\sum_j | b_j |^2 \leq \sup_{\partial D}\sum_j | b_j |^2 = \sup_{\partial D}\sum_j | b_j\sqrt{B} |^2 \leq 1.$$

□

Lemma 4. *Let $v = (v_1, \cdots, v_n)$ and $w = (w_1, \cdots, w_n) \in \boldsymbol{C}^n$. Then*

$$\textstyle\sum_{jk} | v_j w_k - v_k w_j |^2 \leq 2 \| v \|^2 \| w \|^2.$$

Lemma 5. *Let c_i $(i = 1,2)$ and d_i $(i = 1,2,3,4)$ be functions on D. If we set $X_j = c_1 a_j + c_2 b_j$ and $Y_j = d_1 a_j + d_2 b_j + d_3 a_j' + d_4 b_j'$ $(1 \leq j \leq n)$, then*

$$\begin{aligned}&\textstyle\sum_{jk} | X_j Y_k - Y_j X_k |^2\\ &\leq 20(| c_1 |^2 + | c_2 |^2)\{| d_1 |^2 + | d_2 |^2\\ &+ (| d_3 |^2 + | d_4 |^2)(\| a' \|^2 + \| b' \|^2)\}.\end{aligned}$$

Proof. By the Binet-Cauchy formula we have

$$X_j Y_k - Y_j X_k = \text{determinant of}$$

$$\begin{Bmatrix} c_1 & c_2 & 0 & 0 \\ d_1 & d_2 & d_3 & d_4 \end{Bmatrix} \begin{Bmatrix} a_j & a_k \\ b_j & b_k \\ a_j' & a_k' \\ b_j' & b_k' \end{Bmatrix}.$$

By Schwarz's inequality and Lemma 3 and Lemma 4 we obtain

$$\begin{aligned}\textstyle\sum_{jk} | X_j Y_k - Y_j X_k |^2 &\leq 5 \textstyle\sum_{jk}\{| c_1 d_2 - c_2 d_1 |^2\\ &| a_j b_k - a_k b_j |^2 + | c_1 d_3 |^2 | a_j a_k' - a_k a_j' |^2\\ &+ |c_1 d_4|^2 | a_j b_k' - a_k b_j' |^2 + | c_2 d_3 |^2 | b_j a_k' - b_k a_j' |^2\end{aligned}$$

$+ |c_2d_4|^2 \,|\, b_jb_k' - b_kb_j'|^2\} \leq 20\{(|c_1|^2 + |c_2|^2)$

$(|d_1|^2 + |d_2|^2) + |c_1|^2 |d_3|^2 \|a'\|^2$

$+ |c_1|^2 |d_4|^2 \|b'\|^2 + |c_2|^2 |d_3|^2 \|a'\|^2$

$+ |c_2|^2 |d_4|^2 \|b'\|^2\}$ □

6. We will give estimates of $\|u\|^2$, $\|v\|^2$, $\|\partial u\|$ and $\|\partial v\|$.

Proposition 3. *There exists a constant* C *such that*

$$\delta^{16}\|u\|^2,\ \delta^{16}\|v\|^2,\ \delta^{12}\|\partial u\| \text{ and}$$
$$\delta^{12}\|\partial v\| \leq C(\|a'\|^2 + \|b'\|^2 + |B'|^2).$$

Proof. Let $\omega = (\|a'\|^2 + \|b'\|^2 + |B'|^2)^{1/2}$. From (6), it follows that

$$\partial x_j = \{(a', a) + (b', b)\}a_j - \{(a', b) + (b', a)B + (b, a)B'\}b_j + (\|a\|^2 + \|b\|^2)a_j' - \{(a, b) + (b, a)B\}b_j'.$$

Set $X_j = x_j$ and $Y_j = \partial x_j$ in Lemma 4, then $c_1 = d_3 = (\|a\|^2 + \|b\|^2)$, $c_2 = d_4 = -(a, b) + (b, a)B$, $d_1 = (a', a) + (b', b)$ and $d_2 = -(a', b) + (b', a)B + (b, a)B'$.

Since $|c_1| \leq 2$, $|c_2| \leq 2$, $|d_1| \leq 2\omega$, $|d_2| \leq 3\omega$, $|d_3| \leq 2$ and $|d_4| \leq 2$, we obtain

$$\textstyle\sum_{jk} |x_j\partial x_k - x_k\partial x_j|^2 \leq \text{const. } \omega^2.$$

Similarly we have

$$\textstyle\sum_{jk} |y_j\partial y_k - y_k\partial y_j|^2 \leq \text{const. } \omega^2.$$

By (9) and Schwarz's inequality, we have

$$\|u\|^2 = \textstyle\sum_{jk} |u_{jk}|^2 \leq 2\rho^{-4}\sum_{jk}\{|x_j\partial x_k - x_k\partial x_j|^2 + |y_j\partial y_k - y_k\partial y_j|^2\} \leq \text{const. } \delta^{-16}\omega^2.$$

By $|\partial\rho|^2 \leq$ const. ω^2 and Lemma 4, direct computations give

$$\|\partial u\|^2 = \textstyle\sum_{jk} |\partial u_{jk}|^2$$
$$\leq (4\rho^{-6}\cdot 2 + \rho^{-4}\cdot 5)\textstyle\sum_{jk}\{|\partial\rho|^2 |x_j\partial x_k - x_k\partial x_j|^2$$
$$+ |\partial\rho|^2 |y_j\partial y_k - y_k\partial y_j|^2 + |\bar\partial x_j\partial x_k - \bar\partial x_k\partial x_j|^2$$
$$+ |\bar\partial y_j\partial y_k - \bar\partial y_k\partial y_j|^2 + |x_j\partial\bar\partial x_k - x_k\partial\bar\partial x_j|^2$$
$$+ |y_j\partial\bar\partial y_k - y_k\partial\bar\partial y_j|^2 + |y_j\partial y_k - y_k\partial y_j|^2 |B'|^2\}$$
$$\leq \text{const. } \delta^{-24}\omega^4.$$

Similarly, estimates of $\|v\|^2$ and $\|\partial v\|^2$ are obtained. □

7. Proof of Theorem 2. If we set

$$w = \|a\|^2 + \|b\|^2 + |B|^2,$$

then we have $\Delta w = 4(\|a'\|^2 + \|b'\|^2 + |B'|^2)$. If we apply Lemma 2 to the function w, then

$$\left(\iint_D |\varphi|^2 (\Delta w)\log\frac{1}{|z|}\,dxdy\right)^{1/2}$$
$$\leq (2\pi e\|w\|_\infty)^{1/2}\|\varphi\|_2 \leq (2\pi e\cdot 3)^{1/2}.$$

By Proposition 3 and Lemma 1, Theorem 2 holds. □

8. Proof of corollary. Let π be the projection from R to D. The function $F = \sum_j |f_j|^2$ is continuous on R, so that its sum converges uniformly on any compact subset of R by Dini's Theorem. Let $\{D_n\}$ be a sequence of discs such that $\bar D_n \subset D_{n+1}$, $\bigcup_n D_n = D$ and there exists no branch point of R above ∂D_n. For each $n \geq 2/\delta$, there exists an $N(n)$ such that $\sum_{j>N(n)} |f_j|^2 \leq n^{-2}$ on the two-sheeted disc $\pi^{-1}(D_n)$, where we have $\sum_{j\leq N(n)} |f_j|^2 \geq (\delta/2)^2$. We assume that $N(n) \leq N(n+1)$. By Theorem 1, there exists $\{g_{nj}\}_{j\leq N(n)}$ functions in $H^\infty(\pi^{-1}(D_n))$ such that $\sum_j f_jg_{nj} = 1$ and $\sum_j |g_{nj}|^2 \leq C(\delta/2)^2$. We set $g_{nj} = 0$ if $j > N(n)$. By Cantor's diagonal process, we may assume that, for any j, the sequence $\{g_{nj}\}$ converges uniformly on any compact subset of R. Let g_j be the limit of $\{g_{nj}\}$. For $m \geq N(k)$ and $n \geq k$, we have

$$|\textstyle\sum_{1\leq j\leq m} f_jg_{nj} - 1| = |\sum_{1\leq j\leq m} f_jg_{nj} - \sum_{j\leq N(n)} f_jg_{nj}|$$
$$\leq 2\textstyle\sum_{j>N(k)} |f_jg_{nj}| \leq (2/k)C(\delta/2)$$

on $\pi^{-1}(D_k)$. Letting $n \to \infty$ and then $k \to \infty$, we have $\sum_j f_jg_j = 1$. □

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