Transcendence of Jacobi's Theta Series

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1. Introduction. This note will show that a recent theorem of Nesterenko ([6], [7]) has an interesting consequence on the independence problem of the values which Jacobi's theta functions take at algebraic numbers as formulated in the theorem stated below.

Let us recall some known facts. As usual we set

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$
$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$
$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$. Mahler [5] proves that $E_2(q)$, $E_4(q)$, $E_6(q)$ are algebraically independent over $\boldsymbol{C}(q)$. Letting "$'$" denote the derivation $q\dfrac{d}{dq}$, we have

$$E_2' = \frac{1}{12}(E_2^2 - E_4),\ E_4' = \frac{1}{3}(E_2E_4 - E_6),$$
$$E_6' = \frac{1}{2}(E_2E_6 - E_4^2).$$

(cf. Lang [4]). By the use of

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2),\ J = \frac{E_4^3}{\Delta},$$

the modular function $j(\tau)$ is described as $j(\tau) = J(q)$, where $q = e^{2\pi i \tau}$ (cf. Apostol [2]). By the equalities

$$\frac{J'}{J} = -\frac{E_6}{E_4},\ \frac{J''}{J'} = -\frac{1}{6}E_2 + \frac{2}{3}\frac{E_6}{E_4} + \frac{1}{2}\frac{E_4^2}{E_6},$$

$E_6 \in \boldsymbol{Q}(J, J', J'')$ and hence $\boldsymbol{Q}(E_2, E_4, E_6) = \boldsymbol{Q}(J, J', J'')$.

Nesterenko's theorem ([6], [7]). *If* $\alpha \in \boldsymbol{C}$, $0 < |\alpha| < 1$ *then*

$$\text{trans.deg}_{\boldsymbol{Q}}\boldsymbol{Q}(\alpha, E_2(\alpha), E_4(\alpha), E_6(\alpha)) \geq 3.$$

Corollary. *If* $\alpha \in \bar{Q}$, $0 < |\alpha| < 1$ *then each of the following sets*
1) $E_2(\alpha), E_4(\alpha), E_6(\alpha)$ 2) $J(\alpha), J'(\alpha), J''(\alpha)$
is algebraically independent.

Here we investigate the values which Jacobi's theta functions

$$\theta_3 = 1 + 2\sum_{n=1}^{\infty} q^{n^2},\ \theta = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2},$$
$$\theta_2 = 2q^{1/4}\sum_{n=1}^{\infty} q^{n(n-1)}$$

take at algebraic numbers. Now we have

Theorem. *Let* $y = y(q)$ *denote any one of* θ_3, θ, θ_2. *If* $\alpha \in \bar{\boldsymbol{Q}}$, $0 < |\alpha| < 1$ *then* $y(\alpha)$, $y'(\alpha)$, $y''(\alpha)$ *are algebraically independent.*

We remark that y is known to satisfy an algebraic differential equation of the third order defined over $\boldsymbol{Q}$ (cf. Jacobi [3]).

2. Proof of the theorem. Let

$$E(q) = q^{1/24}\sum_{n=1}^{\infty}(1 - q^n).$$

This is known to satisfy $E(q)^{24} = \Delta(q)$.

Lemma.
$\theta_3 = E(q)^{-2}E(q^2)^5E(q^4)^{-2}$, $\theta = E(q)^2E(q^2)^{-1}$,
$\theta_2 = 2E(q^2)^{-1}E(q^4)^2$.

Proof. Taking $z = 1$ in Jacobi's triple product identity (Apostol [4, Th. 14.6])

$$\sum_{n=-\infty}^{\infty} q^{n^2}z^n = \prod_{n=1}^{\infty}(1 - q^{2n})(1 + q^{2n-1}z)(1 + q^{2n-1}z^{-1}),$$

we have

$$\theta_3 = \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty}(1 - q^{2n})(1 + q^{2n-1})^2$$
$$= \prod_{n=1}^{\infty}\frac{(1 - q^{2n})(1 + q^n)^2}{(1 + q^{2n})^2}$$
$$= \prod_{n=1}^{\infty}\frac{(1 - q^{2n})^3(1 - q^{2n})^2}{(1 - q^{4n})^2(1 - q^n)^2}.$$

Hence the first equality follows. In the case where $z = -1$ we have

$$\theta = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \prod_{n=1}^{\infty}(1 - q^{2n})(1 - q^{2n-1})^2$$
$$= \prod_{n=1}^{\infty}\frac{(1 - q^{2n})(1 - q^n)^2}{(1 - q^{2n})^2} = \prod_{n=1}^{\infty}\frac{(1 - q^n)^2}{1 - q^{2n}}$$

and the second. If we let $z = q^{-1}$ then

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$$\sum_{n=-\infty}^{\infty} q^{n(n-1)} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-2})$$
$$= 2 \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})^2$$
$$= 2 \prod_{n=1}^{\infty} \frac{(1 - q^{4n})(1 - q^{4n})}{1 - q^{2n}}.$$

The left hand side is easily seen to be $2 \sum_{n=1}^{\infty} q^{n(n-1)}$, which implies the third equality and completes the proof.

Now let us turn to the proof of the theorem. We first note that $K = \mathbf{Q}(J(q), J'(q), J''(q)) = \mathbf{Q}(E_2(q), E_4(q), E_6(q))$ is a differential field. Since $J(q^2)$ is algebraic over $\mathbf{Q}(J(q))$, hence over K, so is each of $J(q^2)$, $J'(q^2) = q^2 \frac{dJ(q^2)}{dq^2} = \frac{1}{2} J(q^2)'$, $J''(q^2)$. Therefore each of $E_2(q^2)$, $E_4(q^2)$, $E_6(q^2)$ is algebraic over K. This shows $\Delta(q^4)$ as well as $\Delta(q^2)$ algebraic over K. It follows that each of $E(q)$, $E(q^2)$, $E(q^4)$ is algebraic over K, and so is each of $\theta_3(q)$, $\theta(q)$, $\theta_2(q)$. Let y be any one of $\theta_3(q)$, $\theta(q)$, $\theta_2(q)$. The functions y, y', y'' are all algebraic over K. By the corollary in the introduction $\mathbf{Q}(E_2(\alpha), E_4(\alpha), E_6(\alpha))$ has transcendence degree 3. According to Weil [9, p. 28, Th. 3], the specialization (guaranteed by the convergence of each function)

$$(E_2, E_4, E_6, y, y', y'') \rightarrow$$
$$(E_2(\alpha), E_4(\alpha), E_6(\alpha), y(\alpha), y'(\alpha), y''(\alpha))$$

is generic over $\mathbf{Q}$, that is, $\mathbf{Q}(E_2, E_4, E_6, y, y', y'')$ and $\mathbf{Q}(E_2(\alpha), E_4(\alpha), E_6(\alpha), y(\alpha), y'(\alpha), y''(\alpha))$ are isomorphic. According to Nishioka [8], y, y', y'' are algebraically independent over $\mathbf{Q}$, which shows the function values $y(\alpha)$, $y'(\alpha)$, $y''(\alpha)$ algebraically independent and completes the proof.

Remark 1. As a corollary to Neterenko's theorem, we obtain the following: If $\alpha \in \mathbf{C}$, $0 < |\alpha| < 1$ then

$$\text{trans.deg}_{\mathbf{Q}}\mathbf{Q}(\alpha, E_2(\alpha), E'(\alpha), E''(\alpha)) \geq 3.$$

In fact, let $F = \mathbf{Q}(\alpha, E(\alpha), E'(\alpha), E''(\alpha))$. Noting $E(\alpha) \neq 0$, and

$$\frac{E'(q)}{E(q)} = \frac{1}{24}\left(1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}\right) = \frac{1}{24} E_2(q),$$

we have $E_2(\alpha)$, $E_2'(\alpha) \in F$, hence $E_4(\alpha) \in F$. From

$$E(\alpha)^{24} = \frac{1}{1728} (E_4(\alpha)^3 - E_6(\alpha)^2)$$

it follows that $E_6(\alpha)$ is algebraic over F. This implies

$$\text{trans.deg}_{\mathbf{Q}}F \geq \text{trans.deg}_{\mathbf{Q}}\mathbf{Q}(\alpha, E_2(\alpha), E_4(\alpha), E_6(\alpha)) \geq 3.$$

Remark 2. After having submitted the paper we were informed by D. Bertrand that in "Theta functions and transcendence" to appear in The Ramanujan J. Math. he proved a stronger result: $\text{trans. deg}_{\mathbf{Q}}\mathbf{Q}(\alpha, y(\alpha), y'(\alpha), y''(\alpha)) \geq 3$ for $\alpha \in \mathbf{C}$ with $0 < |\alpha| < 1$, the proof of which depends upon a series of explicit relations between modular functions and theta functions. The method utilized in this paper is simpler than his.

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