The Diophantine Equation $a^x + b^y = c^z$. III

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§1. Introduction. In our previous papers [5] and [6], we considered the following conjecture when $(p, q, r) = (2, 2, 3)$ and $(2, 2, 5)$, respectively.

Conjecture. *If a, b, c, p, q, r are fixed positive integers satisfying $a^p + b^q = c^r$ with $p, q, r \geq 2$ and $(a, b) = 1$, then the Diophantine equation*

(1) $$a^x + b^y = c^z$$

has only the positive integral solution $(x, y, z) = (p, q, r)$.

In this paper, we consider the above Conjecture when $p = 2$, $q = 2$ and r is an odd prime. Using known results about the Thue equation and the Baker theory, we show that if c or r is sufficiently large, then it holds for a, b, c satisfying certain conditions as specified in Theorem in §2.

§2. Theorem. We first prepare some lemmas.

In the same way as in the proof of Lemma 1 in [5], we obtain the following:

Lemma 1. *The integral solutions of the equation $a^2 + b^2 = c^r$ with $(a, b) = 1$ and r odd prime are given by*

$$a = \pm u \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j} u^{r-(2j+1)} v^{2j},$$

$$b = \pm v \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j+1} u^{r-(2j+1)} v^{2j},$$

$c = u^2 + v^2$, *where u, v are integers such that $(u, v) = 1$ and $u \not\equiv v \pmod 2$.*

In the following, we consider the case $u = m$, $v = 1$; i.e.

(2) $$a = m \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j} m^{r-(2j+1)},$$

$$b = \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j+1} m^{r-(2j+1)}, \quad c = m^2 + 1$$

and

$$m \text{ is even.}$$

Lemma 2. *Let r be an odd prime. Let a, b, c be positive integers satisfying* (2) *and $\left(\frac{a}{b}\right) = -1$, where $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol. If the Diophantine equation* (1) *has positive integral solutions (x, y, z), then x and y are even.*

Proof. Since $a^2 + b^2 = c^r$, we have $\left(\frac{c}{b}\right)^r = \left(\frac{c}{a'}\right)^r = 1$, so $\left(\frac{c}{b}\right) = \left(\frac{c}{a'}\right) = 1$ with $a = ma'$. Since $\left(\frac{a}{b}\right) = -1$, x must be even from (1).

If $r \equiv 1 \pmod 4$, then we have $b \equiv 1 \pmod 8$. Thus we have $\left(\frac{m}{b}\right) = 1$. In fact, putting $m = 2^s t$ ($s \geq 1$ and t is odd), $\left(\frac{m}{b}\right) = \left(\frac{2^s}{b}\right)\left(\frac{t}{b}\right) = \left(\frac{t}{b}\right) = \left(\frac{b}{t}\right) = \left(\frac{1}{t}\right) = 1$. Hence we have $-1 = \left(\frac{a}{b}\right) = \left(\frac{m}{b}\right)\left(\frac{a'}{b}\right) = \left(\frac{a'}{b}\right) = \left(\frac{b}{a'}\right)$, which implies that y is even from (1).

If $r \equiv -1 \pmod 4$, then we have $b \equiv -1 \pmod 4$. Since $x \geq 2$, we have $(-1)^y \equiv 1 \pmod 4$ from (1). Thus y is even.

Remark. We checked that the assumption $\left(\frac{a}{b}\right) = -1$ holds for a, b, c in (2) when $r = 3,5,7$ respectively (cf. Lemma 2 in [5] and Lemma 2 in [6]).

In the same way as in the proof of Lemma 3 in [6], we obtain the following:

Lemma 3. *Let r be an odd prime, and let a, b, c be positive integers satisfying $a^2 + b^2 = c^r$ and $(a, b) = 1$. Suppose that there is an odd prime l such that $ab \equiv 0 \pmod l$ and $e \equiv 0 \pmod r$, where e is the order of c modulo l. If the Diophantine equation* (1) *has positive integral solutions (x, y, z) under these conditions, then we have $z \equiv 0 \pmod r$.*

We use the following known Propositions 1,2,3,4 to show Lemma 4.

Proposition 1 (Lebesgue [3]). *The Diophantine equation $x^2 + 1 = y^n$ has no positive integral solutions x, y, n with $n \geq 2$.*

Proposition 2 (Le [2]). *Let X, Y be non-zero integers such that $(X, Y) = 1$ and $2 \mid XY$. Let*

$\varepsilon = X + Y\sqrt{-1}$ *and* $\bar{\varepsilon} = X - Y\sqrt{-1}$. *If*

$$\left|\frac{\varepsilon^n - \bar{\varepsilon}^n}{\varepsilon - \bar{\varepsilon}}\right| \le n$$

for some integer n, *then* $n < 8\cdot 10^6$.

Proposition 3 (Bugeaud and Győry [1]). *Let* $f(x, y)$ *be an irreducible binary form with integer coefficients and degree* $d \ge 3$. *Let* m *be a non-zero integer with* $|m| \le M\ (\ge e)$. *All integral solutions* x, y *of the Thue equation*

$$f(x, y) = m$$

satisfy

$$\log\max(|x|, |y|) < 3^{3(d+9)} d^{18(d+1)} H^{2d-2} \cdot (\log H)^{2d-1} \log M,$$

where H *is the maximum absolute value of the coefficients of* $f(x, y)$.

Proposition 4 ([4]). *For any positive integer* n *and any complex numbers* α, β, *we have*

$$\alpha^n + \beta^n = \sum_{j=0}^{[n/2]} (-1)^j \left[\begin{matrix} n \\ j \end{matrix}\right] (\alpha+\beta)^{n-2j}(\alpha\beta)^j,$$

where $[n/2]$ *is the greatest integer not greater than* $n/2$ *and*

$$\left[\begin{matrix} n \\ j \end{matrix}\right] = \frac{(n-j-1)!\,n}{(n-2j)!\,j!} \text{ is an integer } (0 \le j \le [n/2]).$$

Lemma 4. *Let* r *be an odd prime, and let* a, b, c *be positive integers satisfying* (2). *Let* b *be a prime power. If* $r > 8\cdot 10^6$ *or* $\log c > 10^{10^{14}}$, *then the Diophantine equation*

$$a^{2X} + b^{2Y} = c^{rZ}$$

has only the positive integral solution $(X, Y, Z) = (1,1,1)$.

Proof. It follows from Lemma 1 that we have

$$a^X = \pm u \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j} u^{r-(2j+1)} v^{2j},$$

$$b^Y = \pm v \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j+1} u^{r-(2j+1)} v^{2j},$$

$c^Z = u^2 + v^2$, where $(u, v) = 1$, u is even and v is odd, since b is odd.

Since b is a prime power and $\left(v, \pm \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j+1} u^{r-(2j+1)} v^{2j}\right)$, we see that

$$v = \pm 1, \quad \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j+1} u^{r-(2j+1)} v^{2j} = \pm b^Y, \tag{3}$$

or

$$v = \pm b^Y, \quad \sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{2j+1} u^{r-(2j+1)} v^{2j} = \pm 1. \tag{4}$$

We first consider (3). Then we have

$$u^2 + 1 = c^Z,$$

which has only the solution $Z = 1$ from Proposition 1. Thus since $c = m^2 + 1$, we have $u = \pm m$, so $Y = 1$, $X = 1$.

We next consider (4). The second equation in (4) has no non-trivial solutions if $r = 3, 5$ (cf. Lemma 5 in [5] and Lemma 5 in [6]). Thus we may suppose $r \ge 7$.

Let $\varepsilon = u + v\sqrt{-1}$ and $\bar{\varepsilon} = u - v\sqrt{-1}$. Then we have from (4)

$$\frac{\varepsilon^r - \bar{\varepsilon}^r}{\varepsilon - \bar{\varepsilon}} = \pm 1. \tag{5}$$

Therefore it follows from Proposition 2 that $r < 8\cdot 10^6$.

We next show that $\log c < 10^{10^{14}}$. Let $f(X, Y) = \sum_{j=0}^{(r-1)/2} \left[\begin{matrix} r \\ j \end{matrix}\right] X^{(r-1)/2-j} Y^j$. By Proposition 4, $f(X, Y) \in \mathbf{Z}[X, Y]$ is a homogeneous polynomial of degree $(r-1)/2 \ge 3$. Since

$$\left[\begin{matrix} r \\ 0 \end{matrix}\right] = 1,\ \left[\begin{matrix} r \\ (r-1)/2 \end{matrix}\right] = r,\ r \,\Big|\, \left[\begin{matrix} r \\ j \end{matrix}\right], \quad j = 1,2,\dots,(r-3)/2,$$

we see from Eisenstein's theorem that $f(X, Y)$ is irreducible over $\mathbf{Q}$. By (5), we have $f(-4v^2, c^Z) = \pm 1$, since $\varepsilon - \bar{\varepsilon} = 2v\sqrt{-1}$ and $\varepsilon\bar{\varepsilon} = u^2 + v^2 = c^Z$. Note that the height H of $f(X, Y)$ satisfies $H = \max_{0 \le j \le (r-1)/2} \left[\begin{matrix} r \\ j \end{matrix}\right] < 2^{r-1}$. Hence it follows from Proposition 3 that

$$\log c < \log\max(4v^2, c^z) < 3^{3(d+9)} d^{18(d+1)} (2^{r-1})^{2d-2} (\log 2^{r-1})^{2d-1}$$

with $d = (r-1)/2$.

Substituting the upper bound of r into the above, we obtain $\log c < 10^{10^{14}}$, as desired.

Combining Lemmas 2,3 with Lemma 4, we obtain the following:

Theorem. *Let* r *be an odd prime, and let* a, b, c *be positive integers satisfying* (2) *and* $\left(\frac{a}{b}\right) = -1$. *Let* b *be a prime power. Suppose that there is an odd prime* l *such that* $ab \equiv 0 \pmod{l}$ *and* $e \equiv 0 \pmod{r}$, *where* e *is the order of* c *modulo* l. *If* $r > 8\cdot 10^6$ *or* $\log c > 10^{10^{14}}$, *then the Diophantine equation* $a^x + b^y = c^z$ *has only the positive integral solution* $(x, y, z) = (2,2,r)$.

References

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