Einstein Normal Homogeneous Riemannian Manifold

By Joon-Sik PARK

Department of Mathematics, Pusan Univesity of Foreign Studies, Korea

(Communicated by Heisuke HIRONAKA, M. J. A., Oct. 14, 1996)

In this paper, we get a necessary and sufficient condition for certain normal homogeneous Riemannian manifolds with two irreducible summands by the isotropic representation to be Einstein. And then, we give such an example.

Let G be a compact connected semi-simple Lie group and H a closed subgroup. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras of G and H. Let B be the Killing form of $\mathfrak{g}$. Let g_0 be the normal homogeneous metric in G/H which is induced from $Q(:= -B)$. We consider the $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with $Q(\mathfrak{h}, \mathfrak{m}) = 0$. Let $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ be a Q-orthogonal $Ad(H)$-invariant decomposition such that $Ad(H)|_{\mathfrak{m}_i}$ is irreducible for $i = 1,2$ and assume that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are inequivalent irreducible $Ad(H)$-representation spaces such that
(1) $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$ and $[\mathfrak{m}_2, \mathfrak{m}_2] \subset (\mathfrak{h} + \mathfrak{m}_1) =: \mathfrak{k}$.
Let K be a closed connected subgroup with $H \subsetneq K \subsetneq G$ which has the subalgebra $\mathfrak{k}$ as its Lie algebra.

In this paper, we assume $(G/H, g_o)$ is a compact normal homogeneous Riemannian manifolds satisfying condition (1). The space of G-invariant symmetric covariant 2-tensors on G/H is given by $\{x_1 Q|_{\mathfrak{m}_1} + x_2 Q|_{\mathfrak{m}_2} \mid x_1, x_2 \in R\}$. The Ricci tensor ρ of G-invariant Riemannian metric on G/H is a G-invariant symmetric convariant 2-tensor on G/H, and we idintify ρ with an $Ad(H)$-invariant symmetric bilinear form on $\mathfrak{m}$. Thus ρ is written as $\rho = r_1 Q|_{\mathfrak{m}_1} + r_2 Q|_{\mathfrak{m}_2}$ for some $r_1, r_2 \in R$.

Now, we compute components of Ricci tensor ρ of $(G/H, g_o)$ explicitly. Let $d_i = dim_R \mathfrak{m}_i (i = 1,2)$. Let $\{e_\alpha\}$ be a Q-orthogonal basis adapted to the decomposition of $\mathfrak{m}$, i.e., each $e_\alpha \in \mathfrak{m}_i$ for some $i \in \{1,2\}$, and $\alpha < \beta$ if $e_\alpha \in \mathfrak{m}_1$ and $e_\beta \in \mathfrak{m}_2$. Next set $A^\gamma_{\alpha\beta} = Q([e_\alpha, e_\beta], e_\gamma)$, so that $[e_\alpha, e_\beta]_\mathfrak{m} = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$, and set $\begin{bmatrix} k \\ ij \end{bmatrix} := \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken all over indices α, β, γ, with $e_\alpha \in \mathfrak{m}_i, e_\beta \in \mathfrak{m}_j, e_\gamma \in \mathfrak{m}_k$ $(1 \le i, j, k \le 2)$, and $[\ ,\ ]_\mathfrak{m}$ denotes the $\mathfrak{m}$-component. Then, $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the Q-orthonormal bases chosen for $\mathfrak{m}_1$, $\mathfrak{m}_2$, and

(2) $$\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}.$$

Lemma 1. *The components r_1, r_2 of Ricci tensor ρ on $(G/H, g_0)$ are given*

(3) $$r_k = \frac{1}{2} - \frac{1}{4d_k} \sum_{j,i} \begin{bmatrix} k \\ ji \end{bmatrix} \quad (k = 1,2).$$

Proof. Let $\{e_j^{(k)}\}_{j=1}^{d_k}$ be Q-othonormal basis on $\mathfrak{m}_k$ $(k = 1,2)$. The Ricci tensor ρ on $(G/H, g_o)$ is given by the following (cf. [1], pp. 184–185):

$$\rho(X, X) = -\frac{1}{2}\sum_\alpha Q([X, e_\alpha]_\mathfrak{m}, [X, e_\alpha]_\mathfrak{m}) + \frac{1}{2}Q(X, X) + \frac{1}{4}\sum_{\beta,\alpha} Q([e_\beta, e_\alpha]_\mathfrak{m}, X)^2$$

for $X \in \mathfrak{m}$. From this equation, we have

$$r_k = r(e_l^{(k)}, e_l^{(k)}) = \frac{1}{2} - \frac{1}{2}\sum_{j,i}\sum_s Q([e_l^{(k)}, e_s^{(i)}]_{\mathfrak{m}_j}, [e_l^{(k)}, e_s^{(i)}]_{\mathfrak{m}_j}) + \frac{1}{4}\sum_{j,i}\sum_{s,t} Q([e_s^{(j)}, e_t^{(i)}]_{\mathfrak{m}_k}, e_l^{(k)})^2.$$

As we remarked above,

$$d_k r_k = \sum_{\ell=1}^{d_k} r(e_\ell^{(k)}, e_\ell^{(k)}) = \frac{d_k}{2} - \frac{1}{4}\sum_{j,i} \begin{bmatrix} j \\ ki \end{bmatrix}.$$

Q.E.D.

Homogeneous space K/H in $H \subsetneq K \subsetneq G$ need not be effective in general. So let K' be the quotient of K acting effectively on K/H. We also assume that K' is semi-simple and $cQ|_{\mathfrak{k}'} = Q_{\mathfrak{k}'}$ for some $c > 0$, where $Q_{\mathfrak{k}'}$ is the negative of the Killing form of $\mathfrak{k}'$.

By our assumption (1), we have

(4) $$\begin{bmatrix} 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 22 \end{bmatrix} = 0.$$

We obtain

(5) $$\begin{bmatrix} 2 \\ 12 \end{bmatrix} = (1 - c)d_1,$$

since

$$\begin{aligned}\begin{bmatrix}2\\12\end{bmatrix} &= -\sum_{e_\alpha\in\mathfrak{m}_1} tr_{\mathfrak{m}_2}(pr_{\mathfrak{m}_2}(ad\, e_\alpha))^2\\ &= \sum_{e_\alpha\in\mathfrak{m}_1}\{-tr_{\mathfrak{g}}(ad\, e_\alpha)^2 + tr_{\mathfrak{k}}(ad\, e_\alpha)^2\}\\ &= \sum_{e_\alpha\in\mathfrak{m}_1}\{-tr_{\mathfrak{g}}(ad\, e_\alpha)^2 + tr_{\mathfrak{k}'}(ad\, e_\alpha)^2\}\\ &= \sum_{e_\alpha\in\mathfrak{m}_1}\{Q(e_\alpha, e_\alpha) - Q_{\mathfrak{k}'}(e_\alpha, e_\alpha)\}\\ &= (1-c)d_1\end{aligned}$$

by (4).

Thus, from (4), (5) and Lemma 1, we obtain

Theorem 2. *Assume that G is a compact connected semisimple Lie group, $\mathfrak{m}$ decomposes into two inequivalent irreducible summands which satisfy condition* (1), *and that $\mathfrak{k}' := \mathfrak{h} + \mathfrak{m}_1$ is a subalgebra with $Q_{\mathfrak{k}'} = c\, Q|_{\mathfrak{k}'}$. Then $(G/H, g_0)$ is Einstein if and only if $d_2 = 2d_1$. Moreover, if g_o in $(G/H, g_o)$ is Einstein, then $g_o = \frac{(1+c)}{4}\rho$.*

Example. We consider the case when $G = SO(2n+m)$, $K = SO(2n) \times SO(m)$ and $H = U(n) \times SO(m)$, where $n \geqq 3$, $m \geqq 2$. Note that the imbedding of $U(n)$ into $SO(2n)$ is given by

$$A + \sqrt{-1}B \to \begin{pmatrix} A & B\\ -B & A\end{pmatrix}.$$

The spaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are given by

$$\mathfrak{m}_1 = \left\{\begin{pmatrix} X & Y & 0\\ Y & -X & 0\\ 0 & 0 & 0\end{pmatrix} \mid X, Y \in \mathfrak{so}(n)\right\},$$

$$\mathfrak{m}_2 = \left\{\begin{pmatrix} 0 & Z\\ -{}^tZ & 0\end{pmatrix} \mid Z \text{ is a real } 2n\times m \text{ matrix}\right\}.$$

$\mathfrak{m}_1$ is $Ad(H)$-irreducible (cf. [2]).

Note that $\bar{H} := SO(n)\cdot U_1 (\subset H)$ acts on $\mathfrak{m}_2$ by

$$\begin{pmatrix}\cos\theta\cdot A & \sin\theta\cdot A & 0\\ -\sin\theta\cdot A & \cos\theta\cdot A & 0\\ 0 & 0 & B\end{pmatrix}\begin{pmatrix}0 & 0 & C\\ 0 & 0 & D\\ -{}^tC & -{}^tD & 0\end{pmatrix}\begin{pmatrix}\cos\theta\cdot A & \sin\theta\cdot A & 0\\ -\sin\theta\cdot A & \cos\theta\cdot A & 0\\ 0 & 0 & B\end{pmatrix}^{-1} = \begin{pmatrix}0 & 0 & P\\ 0 & 0 & Q\\ -{}^tP & -{}^tQ & 0\end{pmatrix},$$

where $P = \cos\theta\cdot ACB^{-1} + \sin\theta\cdot ADB^{-1}$, $Q = -\sin\theta\cdot ACB^{-1} + \cos\theta\cdot ADB^{-1}$. Thus we see that $\mathfrak{m}_2$ is an irreducible $Ad(H)$-module. Moreover, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are mutually inequivalent $Ad(H)$-representation spaces and thus the homogeneous manifolds $G/H = SO(2n+m)/(U(n)\times SO(m))$, $n \geqq 3$, $m \geqq 2$, satisfy our assumptions. We also have

(6) $d_1 = (n^2 - n)$, $d_2 = 2nm$, $c = 2/3$.

Thus, from Theorem 2 we get

Theorem 3. *$(SO(2n+m)/(U(n)\times SO(m)), g_o)$, $(n \geqq 3, m \geqq 2)$, are Einstein if and only if $m = (n-1)$. Moreover, if $(SO(2n+m)/U(n)\times SO(m), g_o)$, $(n\geqq 3, m \geqq 2)$, are Einstein, then $g_o = (5/12)\rho$.*

References

[1] A. L. Besse: Einstein Manifolds. Springer Verlag, Berlin (1987).

[2] S. Helgason: Differential Geometry and Symmetric Spaces. Academic Press, New York (1978).

[3] M. Wang and W. Ziller: On normal homogeneous manifolds. Ann. Sci. Ecole Norm. Sup., **18**, 563–633 (1985).

[4] M. Wang and W. Ziller: Existence and non-existence of homogeneous Einstein metrics. Invent. Math., **84**, 177–194 (1986).

[5] J. Wolf: The geometry and structure of isotropy irreducible homogeneous spaces. Acta Math., **120**, 59–148 (1968), Errantum. Acta Math., **152**, 141–142 (1984).