On Hasse Zeta Functions of Enveloping Algebras of Solvable Lie Algebras

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1. Introduction 1.1. In [3], we generalized the definitions of Hasse zeta functions of commutative finitely generated rings over the ring $\boldsymbol{Z}$ of integers, to non-commutative rings. In this paper we compute the Hasse zeta functions of the enveloping algebras of completely solvable Lie algebras having p-mappings.

For a (not necessarily commutative) finitely generated ring A over $\boldsymbol{Z}$, in [3] we defined the Hasse zeta function $\zeta_A(s)$ of A by

$$\zeta_A(s) = \prod_{r\geq 1} \zeta_{A,r}(s)$$

where r runs over integers ≥ 1 and,

$$\zeta_{A,r}(s) = \prod_p \exp \sum_{n=1}^{\infty} \frac{\# \mathfrak{S}_{A,r}(\boldsymbol{F}_{p^n})}{n} (p^{-s})^n$$

where $\mathfrak{S}_{A,r}$ is a certain scheme of finite type over $\boldsymbol{Z}$, p runs over prime numbers, and $\boldsymbol{F}_{p^n}$ is a finite field with p^n elements, so the function $\zeta_{A,r}(s)$ coincides with the product of Weil's zeta functions of $\mathfrak{S}_{A,r} \otimes_{\boldsymbol{Z}} \boldsymbol{F}_p$ [2] for all prime numbers p. We do not review the definition of $\mathfrak{S}_{A,r}$, but what we need in this paper is that for the algebraic closure K of $\boldsymbol{F}_p$, $\mathfrak{S}_{A,r}(K)$ is identified with the set of all isomorphism classes of r-dimensional irreducible representations of A over K, and $\mathfrak{S}_{A,r}(\boldsymbol{F}_{p^n})$ is identified with the $\mathrm{Gal}(K/\boldsymbol{F}_{p^n})$-fixed part of $\mathfrak{S}_{A,r}(K)$.

It has the expression

$$\zeta_A(s) = \prod_M (1 - N(M)^{-s})^{-1}$$

where M runs over the isomorphism classes of finite simple A-modules and $N(M) = \# \mathrm{End}_A(M)$.

1.2. Recall that a solvable Lie algebra $\mathfrak{g}$ over a field is said to be completely solvable if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. (See [1].)

We obtain the following result.

Theorem 1.3. *Let R be a commutative finitely generated ring over $\boldsymbol{Z}$. Let $\mathfrak{g}$ be a Lie algebra over R which is free of finite rank n as an R-module, and let A be the universal enveloping algebra of $\mathfrak{g}$. Assume that for each maximal ideal $\mathfrak{m}$ of R, $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$ is a completely solvable Lie algebra over $R/\mathfrak{m}$ and has a p-mapping (see [1]). Then we have that the function $\zeta_A(s)$ converges, and*

$$\zeta_A(s) = \zeta_R(s-n).$$

Remark 1.3.1. *For $x \in \mathfrak{g}$, let $\mathrm{ad}(x)$ be the inner derivation of $\mathfrak{g}$ defined by x, that is, $\mathrm{ad}(x)(y) = [x, y]$ for $y \in \mathfrak{g}$. For a Lie algebra $\mathfrak{g}$ over a field of characteristic p, $\mathfrak{g}$ has a p-mapping $[p]$ if and only if the following condition holds: For any $x \in \mathfrak{g}$, there exists $y \in \mathfrak{g}$ such that $(\mathrm{ad}(x))^{\mathrm{p}} = \mathrm{ad}(y)$.*

1.4. Example. Every nilpotent Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}^{\mathrm{p}} = 0$ ($\mathfrak{g}^i$ is defined by $\mathfrak{g}^0 = \mathfrak{g}$ and $[\mathfrak{g}^i, \mathfrak{g}] = \mathfrak{g}^{i+1}$ for $i \geq 0$) satisfies the condition of Theorem 1.3. This is because $(\mathrm{ad}(x))^{\mathrm{p}} = 0$ for any $x \in \mathfrak{g}$.

In section 2, we prove Theorem 1.3.

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2. Proof of Theorem 1.3. In this section we prove Theorem 1.3.

2.1. The zeta functions $\zeta_R(s)$, $\zeta_A(s)$ are products of $\zeta_{R/\mathfrak{m}}(s)$, $\zeta_{A/\mathfrak{m}A}(s)$ over all maximal ideals $\mathfrak{m}$ of R, respectively, and $A/\mathfrak{m}A$ are the universal enveloping algebras of $\mathfrak{g}/\mathfrak{m}\mathfrak{g}$ over the finite fields $R/\mathfrak{m}$. So we may assume that R is a finite field k of characteristic p. Let K be the algebraic closure of k.

Theorem 1.3 follows from

Proposition 2.2. *Let $\mathfrak{g}$ be a completely solvable Lie algebra over a finite field k of characteritic $p > 0$ of finite dimension n which has a p-mapping $[p]$. Let A be the universal enveloping algebra of $\mathfrak{g}$, and let $\boldsymbol{F}_q$ be a finite extension of k. Then we have*

$$\# \mathfrak{S}_A^k(\boldsymbol{F}_q) = q^n$$

where $\mathfrak{S}_A = \prod_{r\geq 1}\mathfrak{S}_{A,r}$ and $\mathfrak{S}_A^k(\boldsymbol{F}_q)$ denotes the set of $\boldsymbol{F}_q$-rational points of $\mathfrak{S}_A$ as a k-scheme.

We prove Proposition 2.2 in 2.3 and 2.4.

2.3. There is a surjective map φ from $\mathfrak{S}_A^k(K)$ onto $K^{\oplus n}$, the direct sum of n copies of K. Fix a basis $(e_i)_{1\leq i\leq n}$ of $\mathfrak{g}$. For an element x of

$\mathfrak{S}_A^k(K)$, this map φ is defined by $\varphi(x) = (S(e_i))_{1\le i\le n}$ where S is the "character" of x in the sense of [1] 5.2 (S is a k-linear map $\mathfrak{g}\to K$ such that $S(a)^p = x(a)^p - x(a^{[p]})$ for all $a \in \mathfrak{g}$). φ is surjective by Corollary 3.2 in Chapter 5 in [1].

The map φ is compatible with the action of the Galois group $\mathrm{Gal}(K/k)$.

In what follows, we take a p-mapping of the Lie algebra $\mathfrak{g}$ such that

$h^{[p]} = 0$ for any central element h of $\mathfrak{g}$.

In fact we can take such a p-mapping by Corollary 2.2 (3) in Chapter 2 in [1].

Let

$$\mathfrak{g}^* = \mathrm{Hom}_{k\text{-linear}}(\mathfrak{g}, K).$$

Let

$$X = \{\alpha \in \mathfrak{g}^* ; \alpha([\mathfrak{g}, \mathfrak{g}]) = 0\},$$

and let

$$G = \{\alpha \in X ; \alpha(h^{[p]}) = \alpha(h)^p \text{ for any element } h \in \mathfrak{g}\}.$$

We regard X as the set of all one dimensional representations of $\mathfrak{g}$ over K. Remark that G is a finite abelian group (see Proposition 8.8 (1) in Chapter 5 in [1]).

Notation. For $x \in \mathfrak{S}_A^k(K)$, and for $c \in X$, we denote by $x + c$ the tensor product of x and c (as representation).

We use the following result in [1] Chapter 5, Theorem 8.7.

2.3.1. Let x and x' be elements of $\mathfrak{S}_A^k(K)$. Then $\varphi(x) = \varphi(x')$ if and only if there exists an element α of G such that $x' = x + \alpha$.

2.4. Let

$$\mathrm{Frob}_q : K \to K ; x \mapsto x^q.$$

We denote the map $\mathfrak{S}_A^k(K) \to \mathfrak{S}_A^k(K)$ induced by Frob_q also by Frob_q. By 2.3.1, we have that the image of $x \in \mathfrak{S}_A^k(K)$ in $K^{\oplus n}$ under the map φ is an $\boldsymbol{F}_q$-rational point if and only if there exists an element a of G such that

$$\mathrm{Frob}_q(x) = x + a.$$

For $a \in G$, we put

$$F_a = \{x \in \mathfrak{S}_A^k(K); \mathrm{Frob}_q(x) = x + a\}.$$

The following two lemmas 2.4.1 and 2.4.2 prove Proposition 2.2.

Lemma 2.4.1. *The following equation holds for any $a \in G$.*

$$\# F_a = \# \mathfrak{S}_A^k(\boldsymbol{F}_q).$$

Proof. There exists $b \in X$ which satisfies $a = \mathrm{Frob}_q(b) - b$. This is because K is algebraically closed. For $x \in \mathfrak{S}_A^k(K)$, the condition $x \in F_a$ is equivalent to $\mathrm{Frob}_q(x) = x + \mathrm{Frob}_q(b) - b$, and hence to $\mathrm{Frob}_q(x - b) = x - b$. Hence there is a bijection from F_a into $\mathfrak{S}_A^k(\boldsymbol{F}_q)$ by $x \mapsto x - b$. □

Lemma 2.4.2. *The following equation holds for any $a \in G$.*

$$\# F_a = q^n.$$

To prove Lemma 2.4.2, we use the following Lemma 2.4.3-Lemma2.4.5.

Lemma 2.4.3. *For each $x \in \mathfrak{S}_A^k(K)$, let G_x be the subgroup of G defined by*

$$G_x = \{\alpha \in G ; x + \alpha = x\}.$$

Under the canonical map $\coprod_{a\in G} F_a \to \cup_{a\in G} F_a = \varphi^{-1}(\boldsymbol{F}_q^{\oplus n}) \subset \mathfrak{S}_A^k(K)$, the inverse image of an element x of $\cup_{a\in G} F_a$ is of order $\#(G_x)$.

Proof. For $x \in F_a$ and $a' \in G$, the condition $x \in F_{a+a'}$ is equivalent to the condition $a' \in G_x$. This proves the result. □

Lemma 2.4.4. *For any $x \in \cup_{a\in G} F_a$, the order of $\varphi^{-1}(\varphi(x))$ is $\#(G/G_x)$.*

Proof. $\varphi^{-1}(\varphi(x))$ is the G-orbit of x, and hence its order is equal to $\#(G/G_x)$. □

Lemma 2.4.5. *For $x \in \cup_{a\in G} F_a$ and $x' \in \varphi^{-1}(\varphi(x))$, $G_x = G_{x'}$.*

Proof. This follows from the fact that x' belongs to the G-orbit of x and G is commutative. □

Now we prove Lemma 2.4.2. By Lemma 2.4.3–Lemma2.4.5, the inverse image of any element of $\boldsymbol{F}_q^{\oplus n}$ under the map $\varphi : \coprod_{a\in G} F_a \to \boldsymbol{F}_q^{\oplus n}$ is of order $\#(G)$. By Lemma 2.4.1, $\#(F_a)$ is independent of $a \in G$. Hence $\#(G) \cdot \#(F_a) = \#(G) \cdot q^n$. This shows $\#(F_a) = q^n$.

References

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