A Trace Formula for the Picard Group. I

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1. Statement of the result. The aim of the present note is to report the analogue of the Kuznetsov trace formula for the Picard group $\Gamma = PSL(2, \boldsymbol{Z}[i])$ acting discontinuously over the Beltrami model of the three dimensional Lobachevsky geometry (i.e., the hyperbolic upper half-space $\boldsymbol{H}^3$). Our argument is an extension of one of our proofs [3, Section 2.7] of the ordinary Kuznetsov trace formula. We stress that the Picard group has been chosen as a model case. In fact the argument can well be applied to Bianchi groups over arbitrary imaginary quadratic number fields with some extra technical complexities. Our trace formula should have applications to the analytic theory of imaginary quadratic number fields in much the same way as the original Kuznetsov formula has been applied to various important problems in the rational number field. The binary additive divisor problem over imaginary quadratic number fields is one of our targets. It is of particular importance because of its relation with the mean-value problem of Dedekind zeta-functions of respective fields. Such an application will, however, require an enhancement of our formula with the incorporation of Grössencharakters. To these topics and the details of the proof we shall return elsewhere.

To state our trace formula we need some definitions: We denote a point of $\boldsymbol{H}^3$ by $z = (x, y)$ with $x = x_1 + x_2 i$ $(x_1, x_2 \in \boldsymbol{R})$ and $y > 0$. Then the hyperbolic volume element is $d\mu(z) = y^{-3}dx_1dx_2dy$, and the hyperbolic Laplace-Beltrami operator is $\Delta = -y^2((\partial/\partial x_1)^2 + (\partial/\partial x_2)^2 + (\partial/\partial y)^2) + y(\partial/\partial y)$. The set of all Γ-invariant functions over $\boldsymbol{H}^3$ which are square integrable with respect to $d\mu$ over the hyperbolic three-manifold $\mathcal{T} = \Gamma\backslash\boldsymbol{H}^3$ constitutes the Hilbert space $L^2(\mathcal{T}, d\mu)$. The non-trivial discrete spectrum of Δ over $L^2(\mathcal{T}, d\mu)$ is denoted by $\{\lambda_j = 1 + \kappa_j^2 : j = 1,2, \ldots\}$ where $\kappa_j > 0$, and the corresponding orthonormal system of eigenfunctions by $\{\psi_j\}$. We have the Fourier expansion

$$\psi_j(z) = y \sum_{n\in\boldsymbol{Z}[i], n\neq 0} \rho_j(n) K_{i\kappa_j}(2\pi|n|y)e(\langle n, x\rangle).$$

Here K_ν is the K-Bessel function of order ν, $e(a) = \exp(2\pi i a)$, and $\langle n, x\rangle = \mathrm{Re}(n\bar{x})$. We introduce also the Kloosterman sum

$$S(m, n; l) = \sum_{v(\mathrm{mod}\ l), (v,l)=1} e(\langle m, v/l\rangle + \langle n, v^*/l\rangle),\ (l, m, n \in \boldsymbol{Z}[i]),$$

where $vv^* \equiv 1 \bmod l$. Further we shall need the Dedekind zeta-function ζ_K of $\boldsymbol{Q}(i)$ as well as the divisor function $\sigma_\nu(n) = \frac{1}{4}\sum_{d|n}|d|^{2\nu}$ $(n, d \in \boldsymbol{Z}[i])$.

Our trace formula is embodied in

Theorem. *Let us assume that the function* $h(r)$, $r \in \boldsymbol{C}$, *is regular in the horizontal strip* $|\mathrm{Im}\, r| < \frac{1}{2} + \varepsilon$ *and satisfies*

$$h(r) = h(-r),\ h(r) \ll (1 + |r|)^{-3-\varepsilon}$$

with an arbitrary fixed $\varepsilon > 0$. *Then we have, for any non-zero* $m, n \in \boldsymbol{Z}[i]$,

$$\sum_{j=1}^{\infty} \frac{\overline{\rho_j(m)}\rho_j(n)}{\sinh(\pi\kappa_j)} \kappa_j h(\kappa_j) + 2\pi\int_{-\infty}^{\infty} \frac{\sigma_{ir}(m)\sigma_{ir}(n)}{|mn|^{ir}|\zeta_K(1+ir)|^2} h(r)dr$$
$$= (\delta_{m,n} + \delta_{m,-n})\pi^{-2}\int_{-\infty}^{\infty} r^2 h(r)dr + \sum_{l\in\boldsymbol{Z}[i], l\neq 0} |l|^{-2} S(m, n; l)\check{h}(2\pi\varpi)$$

with $\varpi^2 = \overline{mn}/l^2$. *Here* $\delta_{m,n}$ *is the Kronecker delta, and*

$$\check{h}(t) = i\int_{-\infty}^{\infty} \frac{r^2}{\sinh(\pi r)} J_{ir}(t)J_{ir}(\bar{t})h(r)dr$$

with J_ν *being the* J*-Bessel function of order* ν.

2. Sketch of the proof. First we introduce the non-holomorphic Poincaré series over Γ: For $m \in \boldsymbol{Z}[i]$ we put

$$P_m(z, s) = \sum_{\gamma\in\Gamma_t\backslash\Gamma} y(\gamma z)^s \exp(-2\pi|m|y(\gamma z) + 2\pi i\langle m, x(\gamma z)\rangle)\ (z \in \boldsymbol{H}^3, s \in \boldsymbol{C}),$$

where Γ_t is the translation subgroup in Γ (see Sarnak [5]). Expanding this into a double Fourier series with respect to the variables x_1, x_2 we get immediately

$$P_m(z,s) = 2y^s \exp(-2\pi|m|y)\cos(2\pi\langle m,x\rangle) + \frac{1}{2}y^{2-s}\sum_{n\in\mathbf{Z}[i]} e(\langle n,x\rangle)\sum_{l\in\mathbf{Z}[i],l\neq 0}|l|^{-2s} \times S(m,n;l)A_s(m,n;l;y),$$

where

$$A_s(m,n;l;y) = \int_{\mathbf{C}}(|\xi|^2+1)^{-s}\exp\Big(-2\pi iy\langle n,\xi\rangle - \frac{2\pi|m|}{|l|^2(|\xi|^2+1)y} - \frac{2\pi i\langle l^{-2}, m\xi\rangle}{(|\xi|^2+1)y}\Big)d\xi_1 d\xi_2$$

$$(\xi = \xi_1 + i\xi_2).$$

This expansion and an estimate of $S(m,n;l)$ obtainable by the argument due to Gundlach [2] imply that $P_m(z,s)$ is an element of $L^2(\mathcal{F}, d\mu)$ whenever $m\neq 0$ and $\mathrm{Re}\, s > \frac{3}{2}$. Changing the variable by putting $\xi = ue^{i\theta}$ we have also

$$A_s(m,n;l;y) = 2\pi\int_0^\infty \frac{u}{(u^2+1)^s} \times J_0\Big(2\pi u\Big|ny + \frac{\bar{m}}{l^2(u^2+1)y}\Big|\Big) \times \exp\Big(-\frac{2\pi|m|}{|l|^2(u^2+1)y}\Big)du.$$

We then consider the inner-product $[P_m(\cdot, s_1), P_n(\cdot, \overline{s_2})]$ in the space $L^2(\mathcal{F}, d\mu)$, where $mn\neq 0$; and $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2 > \frac{3}{2}$. We have

$$[P_m(\cdot,s_1),P_n(\cdot,\overline{s_2})] = (\delta_{m,n}+\delta_{m,-n})\Gamma(s_1+s_2-2) \times (4\pi|m|)^{2-s_1-s_2} + \pi(|m|/|n|)^{\frac{1}{2}(s_2-s_1)}\sum_{l\in\mathbf{Z}[i],l\neq 0} |l|^{-s_1-s_2}S(m,n;l)B(2\pi\sqrt{mn}/|l|, \vartheta_0; s_1, s_2),$$

where $\vartheta_0 = \arg\varpi$ with ϖ as above, and

$$B(p,\vartheta;s_1,s_2) = \int_0^\infty y^{s_2-s_1-1} \times C\Big(p,\vartheta;\frac{1}{2}(s_1+s_2);y\Big)dy$$

with $|\vartheta|\le\frac{1}{2}\pi$ and

$$C(p,\vartheta;\tau;y) = \int_0^\infty \frac{u}{(u^2+1)^\tau}\exp\Big(-\frac{p(y+y^{-1})}{\sqrt{u^2+1}}\Big) \times J_0\Big(\frac{pu}{\sqrt{u^2+1}}|ye^{i\vartheta}+(ye^{i\vartheta})^{-1}|\Big)du.$$

To separate the variables in this integral we use the Mellin transform of $e^{-a}J_0(ab)$ as a function of $a>0$ (cf. [1, 6.621(1)]), which involves the hypergeometric function. Invoking the Mellin-Barnes formula for the hypergeometric function we are led to the expression

$$\exp(\cdots)J_0(\cdots) = -\frac{1}{2\pi^2}\int_{(\alpha)}\Big(\frac{p(y+y^{-1})}{\sqrt{u^2+1}}\Big)^{-2\eta}\int_{(\beta)} \frac{\Gamma(2\xi+2\eta)\Gamma(-\xi)}{\Gamma(\xi+1)}\Big(\frac{1}{4}u^2\Big(1-\Big(\frac{2\sin\vartheta}{y+y^{-1}}\Big)^2\Big)\Big)^\xi d\xi d\eta$$

where $\alpha>0$, $\beta>0$ are small while satisfying $\alpha+\beta>0$. This double integral is, however, not absolutely convergent. To gain the absolute convergence we shift the contour (β) to (β') with a small $\beta'>0$. The pole at $\xi=0$ contributes a term which does not cause any trouble. We insert the resulting expression into the integral for $C(p,\vartheta;\tau;y)$. The triple integral thus obtained is absolutely convergent provided $\mathrm{Re}\,\tau>1$. We perform the u-integral first, getting

$$C(p,\vartheta;\tau;y) = R - \frac{1}{4\pi^2}\int_{(\alpha)}\int_{(\beta')} \frac{\Gamma(2\xi+2\eta)\Gamma(\tau-\xi-\eta-1)\Gamma(-\xi)}{\Gamma(\tau-\eta)} \times (p(y+y^{-1}))^{-2\eta}\Big(\frac{1}{4}\Big(1-\Big(\frac{2\sin\vartheta}{y+y^{-1}}\Big)^2\Big)\Big)^\xi d\xi d\eta,$$

where R is the contribution of the pole at $\xi=0$. We insert this into the integral for $B(p,\vartheta;s_1,s_2)$. The arising triple integral is absolutely convergent provided $|\mathrm{Re}(s_1-s_2)|<2\alpha$ which we shall assume for a while. We arrange the order of integration by putting the ξ-integral inner, the y-integral middle, and the η-integral outer. To compute the inner integral we expand the factor $(1-(2\sin\vartheta/(y+y^{-1}))^2)^\xi$ into a binomial series. The ξ-integral can be performed inside this expansion; the termwise integration can be accomplished with the aide of Barnes' integral formula involving four Γ-factors. The resulting series converges absolutely, and the y-integral can be performed termwise. Then we find that

$$B(p,\vartheta;s_1,s_2) = -i\frac{\Gamma(s_1+s_2-2)}{2^{s_1+s_2-1}\sqrt{\pi}} \times \int_{(\alpha)}\frac{p^{-2\eta}}{\Gamma\big(\frac{1}{2}(s_1+s_2)-\eta\big)}\times\sum_{\nu=0}^\infty \frac{\Gamma\big(\eta+\frac{1}{2}(s_1-s_2)+\nu\big)\Gamma\big(\eta+\frac{1}{2}(s_2-s_1)+\nu\big)}{\Gamma(\nu+1)\Gamma\big(\eta+\frac{1}{2}(s_1+s_2-1)+\nu\big)} \times (\sin\vartheta)^{2\nu}d\eta.$$

This integral is absolutely convergent, but the whole expression is not absolutely convergent. To attain the same effect as the exchange of the order of the sum and the integral we invoke Gauss' representation of the hypergeometric function as an integral over the unit interval. Then we have an absolutely convergent double integral. The η-integral can be taken inner, and we get, after a change of variable,

$$B(p,\vartheta;s_1,s_2)=8\sqrt{\pi}\frac{p^{1-s_2}\Gamma(s_1+s_2-2)}{2^{s_1+s_2}\Gamma\left(s_1-\frac{1}{2}\right)}$$

$$\times\int_0^\infty u^{2s_1-2}(u^2+1)^{\frac{1}{2}-s_2}\times(u^2+(\cos\vartheta)^2)^{\frac{1}{2}(1-s_1)}$$

$$\times J_{s_1-1}\left(2p\sqrt{u^2+(\cos\vartheta)^2}\right)du.$$

Expressing the factor $(u^2+1)^{\frac{1}{2}-s_2}$ as an inverse Mellin integral, we are led to the situation where we may appeal to an integral formula due to Sonine [1, 6.596(1)]. This procedure yields

$$B(p,\vartheta;s_1,s_2)=\frac{(2p)^{2-s_1-s_2}\Gamma(s_1+s_2-2)}{2i\sqrt{\pi}\Gamma\left(s_1-\frac{1}{2}\right)\Gamma\left(s_2-\frac{1}{2}\right)}$$

$$\times\int_{(\delta)}\Gamma(\xi)\Gamma\left(s_1-\frac{1}{2}-\xi\right)\Gamma\left(s_2-\frac{1}{2}-\xi\right)$$

$$\times J_{\xi-\frac{1}{2}}(2p\cos\vartheta)\left(\frac{p}{\cos\vartheta}\right)^{\xi-\frac{1}{2}}d\xi,$$

where $0<\delta<\min[\mathrm{Re}\,s_1,\mathrm{Re}\,s_2]-\frac{1}{2}$; and the above restriction on s_1, s_2 has been dropped. We replace the J-factor by its defining series expansion, getting an absolutely convergent expression. Thus we have

$$\int_{(\delta)}\cdots=\sum_{\nu=0}^\infty\frac{(-1)^\nu}{\nu!}(p\cos\vartheta)^{2\nu}$$

$$\times\int_{(\delta)}\frac{\Gamma(\xi)\Gamma\left(s_1-\frac{1}{2}-\xi\right)\Gamma\left(s_2-\frac{1}{2}-\xi\right)}{\Gamma\left(\xi+\frac{1}{2}+\nu\right)}p^{2\xi-1}d\xi.$$

The factor $\Gamma(\xi)/\Gamma\left(\xi+\frac{1}{2}+\nu\right)$ can be expressed in terms of the Beta-integral of Euler. The new sum over ν is a constant multiple of the series for cosine, and we get, after a change of variable,

$$\sum_{\nu=0}^\infty\cdots=\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2}\pi}\cos(2p\cos\vartheta\cos\tau)$$

$$\times\int_{(\delta)}\Gamma\left(s_1-\frac{1}{2}-\xi\right)\Gamma\left(s_2-\frac{1}{2}-\xi\right)(p\sin\tau)^{2\xi-1}d\xi d\tau.$$

The inner integral is essentially a value of the K-Bessel function of order s_1-s_2. But we rather appeal to the following integral formula (see [3, Lemma 2.7]):

$$\frac{i}{2\pi^2}\int_{-i\infty}^{i\infty}\lambda\sin(2\pi\lambda)\Gamma(\omega_1+\lambda)\Gamma(\omega_2+\lambda)\Gamma(\omega_3+\lambda)$$

$$\times\Gamma(\omega_1-\lambda)\Gamma(\omega_2-\lambda)\Gamma(\omega_3-\lambda)d\lambda$$

$$=\Gamma(\omega_1+\omega_2)\Gamma(\omega_2+\omega_3)\Gamma(\omega_3+\omega_1),$$

where the path is curved to ensure that the poles $\Gamma(\omega_1+\xi)\Gamma(\omega_2+\xi)\Gamma(\omega_3+\xi)$ lie to the left of the path, and those of $\Gamma(\omega_1+\xi)\Gamma(\omega_2+\xi)\Gamma(\omega_3+\xi)$ to the right, providing parameters ω_1, ω_2, ω_3 are such that the path can be drawn. We put $\omega_1=s_1-1$, $\omega_2=s_2-1$, $\omega_3=\frac{1}{2}-\xi$ assuming $\mathrm{Re}\,\xi=\delta$ is small. Then we are led to

$$\sum_{\nu=0}^\infty\cdots=\frac{8\sqrt{\pi}}{\Gamma(s_1+s_2-2)}\int_{(0)}\lambda\sin(2\pi\lambda)\Lambda(s_1,s_2;\lambda)$$

$$\times\int_0^{\frac{1}{2}\pi}\cos(2p\cos\vartheta\cos\tau)K_{2\lambda}(2p\sin\tau)d\tau d\lambda,$$

where

$$\Lambda(s_1,s_2;\lambda)=\Gamma(s_1-1+\lambda)\Gamma(s_1-1-\lambda)$$
$$\Gamma(s_2-1+\lambda)\Gamma(s_2-1-\lambda).$$

The last inner-integral is expressible in terms of Bessel functions. In fact the formula 6.688(1) of [1] yields, via analytic continuation,

$$\int_0^{\frac{1}{2}\pi}\cdots=\frac{\pi^2}{4\sin(2\pi\lambda)}$$

$$\times[J_{-\lambda}(pe^{i\vartheta})J_{-\lambda}(pe^{-i\vartheta})-J_\lambda(pe^{i\vartheta})J_\lambda(pe^{-i\vartheta})].$$

Collecting these we get an expression of $[P_m(\cdot,s_1),P_n(\cdot,\overline{s_2})]$ involving the function Λ. Equating the result with the spectral decomposition of the inner-product we obtain an identity which is exactly the same as the specialization of the theorem with $h(r)=\pi\Lambda(s_1,s_2;r)\,|\,\Gamma(1+ir)\,|^{-2}$. We then put $s_1=s$, $s_2=2$, and multiply the resulting identity by the factor

$$\mathrm{f}_{(s)}=\frac{1}{8\pi i}\int_0^\infty g(x)\,(x/2)^{1-2s}dx$$

with a suitable g. Further, integrating with respect to s, $\mathrm{Re}\,s=2$, we get the theorem with $h(r)$ being replaced by the integral

$$\int_{(2)}\Gamma(s-1+ir)\Gamma(s-1-ir)f(s)ds$$

$$=\int_0^\infty g(x)K_{2ir}(x)x^{-1}dx\quad(r\in\boldsymbol{R}).$$

By virtue of the dual of the Kontrovitch-Lebedev inversion formula (see [3, Lemma 2.10]) for the K-Bessel transform, we see that g can be chosen so that the last integral coincides with the $h(r)$ given in the theorem. This ends the proof. It should be noted that we need actually a result on the spectral mean square of $\rho_j(n)$'s. For this see [4] although our argument can yield the same assertion as theirs.

Remark. From our intermediate trace formula involving Λ one way deduce the Fourier expansion of the resolvent kernel of the Laplacian Δ which has similar features as Fay's result for Fuchsian groups. A possible alternative way to prove our theorem is to use the Zagier transform of point-pair invariants in the context of $\boldsymbol{H}^3$. Our initial experiment indicates, however, that this leads us to a more complicated situation than the above.

References

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