A Remark on Estimates of Bilinear Forms of Gradients in Hardy Space

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§0. Introduction. In a recent interesting paper [1] L.C. Evans and S. Müller established the estimate of local Hardy space norm of gradients ϕ_{x_1}, ϕ_{x_2}:

(0.1) $$\|\phi\phi_{x_1}\phi_{x_2}\|_{h^1} + \|\phi(\phi_{x_1}^2 - \phi_{x_2}^2)\|_{h^1} \le C(\|\phi_{x_1}\|^2_{L^2(B(0,R))} + \|\phi_{x_2}\|^2_{L^2(B(0,R))})$$

provided that

(0.2) $$-\Delta\psi = \omega \geqq 0 \text{ in } \boldsymbol{R}^2.$$

Here ϕ is in $C_0^\infty(\boldsymbol{R}^2)$ and the constants C and R depends only on ϕ ; h^1 is a local Hardy space defined in §1 and $B(x, R)$ denotes the closed ball of radius R centered at $x \in \boldsymbol{R}^2$. (Another proof based on harmonic analysis is given by Semmes [2].)

This estimate is useful in proving the existence of weak solutions for the initial value problem of the two-dimensional Euler equation when the vorticity of the initial value is nonnegative measure ([1] and Delort [3]). The assumption $\omega \geqq 0$ in (0.2) is essential for the estimate (0.1); in fact, Evans and Müller [1] gave a counterexample for (0.1) when the condition $\omega \geqq 0$ is violated. However, in their example the set where ω is nonnegative may be complicated.

In this paper we give another counterexample for (0.1) even when ω is odd in the second variable x_2 i.e. $\omega(x_1, x_2) = -\omega(x_1, -x_2)$ and $\omega(x_1, x_2) \geqq 0$ for $x_2 \geqq 0$. This suggests that it is difficult to extend weak solutions for the initial-boundary value problem of the Euler equation when the domain is a half space $\boldsymbol{R}^2_+$ even if initial value is nonnegative in $\boldsymbol{R}^2_+$.

To get our counterexample we construct a sequence ψ^ε of form $\psi^\varepsilon(x) = \psi(x/\varepsilon)$. A key observation is the existence of function ψ that satisfies

$$\int_{\boldsymbol{R}^2} \psi^2_{x_1}\, dx \neq \int_{\boldsymbol{R}^2} \psi^2_{x_2}\, dx$$

with $-\Delta\psi = \omega$, where $\omega \in C_0^\infty(\boldsymbol{R}^2)$ is odd in the second variable x_2 and $\omega \geqq 0$ in $\boldsymbol{R}^2_+$, and $\psi \in H^1(\boldsymbol{R}^2)$; $H^1(\boldsymbol{R}^2)$ denotes the Sobolev space, i.e. the space of $f \in L^2(\boldsymbol{R}^2)$ with $f_{x_1}, f_{x_2} \in L^2(\boldsymbol{R}^2)$.

§1. Definition and main theorem. We begin with definition of local Hardy space as in [1].

Definition 1.1. Let η be in $C_0^\infty(\boldsymbol{R}^n)$ with $\mathrm{supp}\eta \subset B(0,1)$ and $\int_{\boldsymbol{R}^n} \eta dx = 1$. For a function f in $L^1_{loc}(\boldsymbol{R}^n)$, f^{**} is defined by

(1.1) $$f^{**}(x) = \sup_{0<r<1} \left| r^{-n} \int_{\boldsymbol{R}^n} \eta\left(\frac{x-y}{r}\right) f(y)\, dy \right|.$$

The local Hardy space $\mathscr{H}^1_{loc}$ is defined by

(1.2) $$\mathscr{H}^1_{loc}(\boldsymbol{R}^n) = \{f \in L^1_{loc}(\boldsymbol{R}^n) \mid f^{**} \in L^1_{loc}(\boldsymbol{R}^n)\}.$$

We recall the normed local Hardy space h^1 defined by

(1.3) $$h^1(\boldsymbol{R}^n) = \{f \in L^1(\boldsymbol{R}^n) \mid f^{**} \in L^1(\boldsymbol{R}^n)\}$$

with the norm

$$\|f\|_{h^1(\boldsymbol{R}^n)} = \|f^{**}\|_{L^1(\boldsymbol{R}^n)}.$$

Definition 1.2. For a function f in $C_0^\infty(\boldsymbol{R}^2)$, we define the operator $(-\Delta)^{-1}$ by

(1.4) $$(-\Delta)^{-1} f(x) = \frac{1}{2\pi} \int_{\boldsymbol{R}^2} f(y) \log|x-y|\, dy.$$

Theorem 1.3. *Let T and S be the spaces of form*

$$T = \{\omega \in C_0^\infty(\boldsymbol{R}^2) \mid \omega(x_1, x_2) \geqq 0 \text{ for } x_2 \geqq 0,\ \omega(x_1, x_2) = -\omega(x_1, -x_2)\},$$
$$S = \{(-\Delta)^{-1}\omega \mid \omega \in T\}.$$

Then there exists a sequence $\{\psi^\varepsilon\}_{0<\varepsilon<1}$ in S such that

$$\sup_{0<\varepsilon<1} \|\psi^\varepsilon\|_{H^1(\boldsymbol{R}^2)} < \infty$$

and

(1.5) $$\lim_{\varepsilon\downarrow 0} \|\phi((\psi^\varepsilon_{x_1})^2 - (\psi^\varepsilon_{x_2})^2\}\|_{h^1(\boldsymbol{R}^2)} = \infty$$

where $\phi \in C_0^\infty(\boldsymbol{R}^2)$, $0 \leqq \phi \leqq 1$, $\phi|_{B(0,1/8)} \equiv 1$ *and* $\mathrm{supp}\phi \subset B(0,1/2)$.

§2. Proof of theorem. At first, we show a fundamental estimate in normed local Hardy space; this is an extension of a result to Evans and Müller [1].

Lemma 2.1. *Assume that f is in $L^1(\boldsymbol{R}^n)$, and $\int_{\boldsymbol{R}^n} f(x)\, dx = \mathrm{C_f} \neq 0$. Let $f^\varepsilon(x) = \dfrac{1}{\varepsilon^n} f\left(\dfrac{x}{\varepsilon}\right)$.*

Then $\| f^{\varepsilon} \|_{L^1} = \| f \|_{L^1} < \infty$, *and*

$$\text{(2.1)} \qquad \lim_{\varepsilon \downarrow 0} \| \phi f^{\varepsilon} \|_{h^1(\boldsymbol{R}^n)} = \infty$$

for a function ϕ *in* $C_0^{\infty}(\boldsymbol{R}^n)$ *with* $0 \leqq \phi \leqq 1$, $\phi|_{B(0,1/8)} \equiv 1$, *and* $\operatorname{supp}\phi \subset B(0,1/4)$.

Proof. L^1-estimate is easily obtained by scaling variables. To show the estimate (2.1), assume that the function η in (1.1) satisfies $0 \leqq \eta \leqq 1$ and $\eta|_{B(0,1/2)} \equiv 1$. Now we estimate the function $(\phi f^{\varepsilon})^{**}$:

$$\text{(2.2)} \quad (\phi f^{\varepsilon})^{**}(x) = \sup_{0<r<1} \left| \frac{1}{r^n} \int_{\boldsymbol{R}^n} \eta\left(\frac{x-y}{r}\right) \phi(y) f^{\varepsilon}(y)\, dy \right| = \sup_{0<r<1} \left| \frac{1}{r^n} \int_{\boldsymbol{R}^n} \eta\left(\frac{x-\varepsilon z}{r}\right) \phi(\varepsilon z) f(z)\, dz \right|.$$

Now take a parameter $R > 0$, and let $r = 4|x|$. Then

$$\text{(2.3)} \quad (\phi f^{\varepsilon})^{**}(x) \geqq \frac{1}{4^n |x|^n} \left| \int_{B(0,R)} \eta\left(\frac{x-\varepsilon z}{4|x|}\right) \phi(\varepsilon z) f(z)\, dz \right| - \frac{1}{4^n |x|^n} \left| \int_{\boldsymbol{R}^n \setminus B(0,R)} \eta\left(\frac{x-\varepsilon z}{4|x|}\right) \phi(\varepsilon z) f(z)\, dz \right| = I_1^{\varepsilon} - I_2^{\varepsilon} \text{ for } \varepsilon R < |x| < 1/4.$$

We show that $\lim_{\varepsilon \downarrow 0} \| I_2^{\varepsilon} \|_{L^1(B(0,1/4))} = 0$ and $\lim_{\varepsilon \downarrow 0} \| I_1^{\varepsilon} \|_{L^1(B(0,1/4))} = \infty$ to complete the proof. Firstly, we estimate the term I_2^{ε}:

$$I_2^{\varepsilon}(x) \leqq \frac{1}{(4\varepsilon R)^n} \int_{\boldsymbol{R}^n \setminus B(0,R)} |f(z)|\, dz = \frac{F(R)}{(4\varepsilon R)^n}$$

$$\text{with } F(R) = \int_{\boldsymbol{R}^n \setminus B(0,R)} |f(z)|\, dz.$$

Since $F(R)$ is continuous, nonincreasing, and $\lim_{R \to \infty} F(R) = 0$, for sufficiently small ε, there exists $R = R(\varepsilon)$ such that

$$\text{(2.4)} \qquad \frac{\{F(R)\}^{1/2}}{R^n} = \varepsilon^n.$$

By (2.4), we get

$$\text{(2.5)} \qquad I_2^{\varepsilon}(x) \leqq \frac{\{F(R)\}^{1/2}}{4^n} \longrightarrow 0 \text{ as } \varepsilon \downarrow 0$$

and get $\lim_{\varepsilon \downarrow 0} \| I_2^{\varepsilon} \|_{L^1(B(0,1/4))} = 0$.

Secondly, we estimate the term I_1^{ε}. As $\left| \frac{x - \varepsilon z}{4|x|} \right| \leqq \frac{|x|}{4|x|} + \frac{\varepsilon R}{4|x|} \leqq \frac{1}{2}$,

$$\text{(2.6)} \quad I_1^{\varepsilon} = \frac{1}{4^n |x|^n} \left| \int_{B(0,R)} \phi(\varepsilon z) f(z)\, dz \right| = \frac{1}{4^n |x|^n} \left| \int_{B(0,R)} f(z)\, dz \right| \text{ for } \varepsilon R \leqq |x| \leqq 1/4.$$

for $\varepsilon R < 1/8$.

Now let $\varepsilon \downarrow 0$. For $\varepsilon R = \{F(R)\}^{1/2n} \longrightarrow 0$ by (2.4),

$$\text{(2.7)} \quad \lim_{\varepsilon \downarrow 0} I_1^{\varepsilon} = \frac{1}{4^n |x|^n} \left| \int_{\boldsymbol{R}^n} f(z)\, dz \right| = \frac{|C_f|}{4^n |x|^n} \text{ for } 0 \leqq |x| \leqq 1/4.$$

Since $\frac{1}{|x|^n}$ is not in $L^1(\boldsymbol{R}^n)$, we get a conclusion. □

Next lemma is important to show Lemma 2.3 which is the key to show Theorem 1.3:

Lemma 2.2. *For any function* ϕ *in* S, *there exists a constant* C *depending only on* ϕ *such that*

$$\text{(2.8)} \qquad |\phi(x)| \leqq \frac{C}{1+|x|}$$

$$\text{(2.9)} \qquad |\phi_{x_j}(x)| \leqq \frac{C}{1+|x|^2}, \; j = 1,2$$

for $x \in \boldsymbol{R}^2$.

Proof. If ϕ is in S, then there exists a function ω in T such that

$$\text{(2.10)} \quad \phi(x) = \frac{1}{2\pi} \int_{\boldsymbol{R}^2} \omega(y) \log|x-y|\, dy.$$

Notice that ω is in $C_0^{\infty}(\boldsymbol{R}^2)$, so there is a constant $R = R(\omega)$ such that $\operatorname{supp}\omega \subset B(0, R)$. Since ω is odd in x_2, we get

$$\text{(2.11)} \quad \phi(x) = \frac{1}{2\pi} \int_{B(0,R)} \omega(y) \log|x-y|\, dy = \frac{1}{2\pi} \int_{B_+(0,R)} \omega(y) \log \frac{|x-y|}{|x-\bar{y}|}\, dy,$$

$$\text{(2.12)} \quad \phi_{x_1}(x) = \frac{1}{2\pi} \int_{B(0,R)} \omega(y) \frac{x_1 - y_1}{|x-y|^2}\, dy = \frac{2x_2}{\pi} \int_{B_+(0,R)} \omega(y) \frac{y_2(x_1-y_1)}{|x-y|^2 |x-\bar{y}|^2}\, dy,$$

$$\text{(2.13)} \quad \phi_{x_2}(x) = \frac{1}{2\pi} \int_{B(0,R)} \omega(y) \frac{x_2 - y_2}{|x-y|^2}\, dy = \frac{1}{\pi} \int_{B_+(0,R)} \omega(y) \frac{y_2\{(x_2-y_2)(x_2+y_2) - (x_1-y_1)^2\}}{|x-y|^2 |x-\bar{y}|^2}\, dy,$$

where $B_+(0, R)$ denotes $B(0, R) \cap \boldsymbol{R}_+^2$ and $\bar{y} = (y_1, -y_2)$. Now we show that ϕ and ϕ_{xj} is bounded in $B(0, 2R)$ and that there exists a constant C such that

$$\text{(2.14)} \qquad |\phi(x)| \leqq C|x|^{-1}, \quad |\phi_{x_j}(x)| \leqq C|x|^{-2} \text{ for } |x| \geqq 2R.$$

The boundedness of ϕ and ϕ_j on $B(0,2R)$ is obtained by the fact that $\log|x|$ and $|x|^{-1}$ are in $L^1_{loc}(\boldsymbol{R}^2)$:

$$\text{(2.15)} \quad |\phi(x)| \leqq \frac{\sup|\omega|}{2\pi} \int_{B(0,R)} |\log|x-y||\, dy$$

$$= C_\omega \int_{B(x,R)} |\log|y|| dy$$
$$\leqq C_\omega \int_{B(0,3R)} |\log|y|| dy$$
$$= C_{\omega,R} < \infty,$$

(2.16) $$|\phi_{x_j}(x)| \leqq \frac{\sup|\omega|}{2\pi}\int_{B(0,R)} \frac{1}{|x-y|} dy$$
$$\leqq C_\omega \int_{B(0,3R)} |y|^{-1} dy$$
$$= C_{\omega,R} < \infty$$

for $|x| \leqq 2R$. (Notice that $|y| \leqq |x-y| + |x| \leqq 3R$.)

Now we show (2.14) to complete the proof. We may assume $x_2 \geqq 0$ to estimate ϕ, because ϕ is odd in x_2. By this assumption, we get the following inequality:

(2.17) $$1 \leqq \frac{|x-\bar{y}|}{|x-y|} \leqq \frac{|x-y|+|y-\bar{y}|}{|x-y|}$$
$$\leqq 1 + \frac{2R}{|x|-R} \leqq 1 + \frac{4R}{|x|}$$

for $|x| \geqq 2R$ and $|y| \leqq R$. The inequality (2.17) leads the estimate of ϕ:

(2.18) $|\phi(x)|$
$$\leqq \frac{1}{2\pi}\log\left(1+\frac{4R}{|x|}\right)\int_{B_+(0,R)} \omega(y)dy$$
$$\leqq \frac{1}{2\pi}\frac{4R}{|x|}\log\left(1+\frac{4R}{|x|}\right)^{\frac{|x|}{4R}} \|\omega\|_{L^1(B_+(0,R))}$$
$$\leqq \frac{2R}{\pi}\|\omega\|_{L^1(\boldsymbol{R}^2)}|x|^{-1}.$$

Notice that the inequality

$$\frac{|x|}{2} \leqq |x-y|, \frac{|x|}{2} \leqq |x-\bar{y}|$$

holds for $|x| \geqq 2R$, $|y| \leqq R$. This inequality leads the estimate of ϕ_{x_j}:

(2.19) $|\phi_{x_1}(x)|$
$$\leqq \frac{2|x|}{\pi}\int_{B_+(0,R)} |\omega(y)| \frac{|y|}{|x-y|^2|x-\bar{y}|} dy$$
$$\leqq \frac{16R\|\omega\|_{L^1}}{\pi|x|^2},$$

(2.20) $|\phi_{x_2}(x)|$
$$\leqq \frac{1}{\pi}\int_{B_+(0,R)} |\omega(y)| \frac{2|y|}{|x-y||x-\bar{y}|} dy$$
$$\leqq \frac{8R\|\omega\|_{L^1}}{\pi|x|^2},$$

Combining the estimate (2.15), (2.16), (2.18), (2.19), and (2.20) leads the conclusion (2.8) and (2.9). □

Now we are ready to show the key lemma.

Lemma 2.3. *There exists a function ϕ in S such that*

(2.21) $$\int_{\boldsymbol{R}^2} \{\phi_{x_1}(x)\}^2 dx \neq \int_{\boldsymbol{R}^2} \{\phi_{x_2}(x)\}^2 dx.$$

Proof. Assume that the conclusion is not true, i.e.

(2.22) $$\int_{\boldsymbol{R}^2} \{\phi_{x_1}(x)\}^2 dx = \int_{\boldsymbol{R}^2} \{\phi_{x_2}(x)\}^2 dx$$

for any ϕ in S. Let ϕ is in S, and let $\phi^h(x) = \phi(x_1-h, x_2)$. Then the function ϕ^h and $\phi+\phi^h$ are in S. In fact, there exists a function ω in T such that $\phi = (-\Delta)^{-1}\omega$, and we can write $\phi^h = (-\Delta)^{-1}\omega^h$ and $\phi+\phi^h = (-\Delta)^{-1}(\omega+\omega^h)$, where $\omega^h(x) = \omega(x_1-h, x_2)$. It is obvious that ω^h and $\omega+\omega^h$ are in T.

By the assumption (2.22), we get

(2.23) $$\int_{\boldsymbol{R}^2} \{\phi_{x_1}(x)\}^2 dx = \int_{\boldsymbol{R}^2} \{\phi_{x_2}(x)\}^2 dx,$$
$$\int_{\boldsymbol{R}^2} \{(\phi^h)_{x_1}(x)\}^2 dx = \int_{\boldsymbol{R}^2} \{(\phi^h)_{x_2}(x)\}^2 dx,$$
$$\int_{\boldsymbol{R}^2} \{(\phi+\phi^h)_{x_1}(x)\}^2 dx = \int_{\boldsymbol{R}^2} \{(\phi+\phi^h)_{x_2}(x)\}^2 dx.$$

Combining the equalities (2.23), we get

(2.24) $$\int_{\boldsymbol{R}^2} \phi_{x_1}(x)(\phi^h)_{x_1}(x)dx$$
$$= \int_{\boldsymbol{R}^2} \phi_{x_2}(x)(\phi^h)_{x_2}(x)dx.$$

Now we integrate the equality (2.24) by h:

(2.25) $$\int_{-\infty}^{\infty}\int_{\boldsymbol{R}^2} \phi_{x_1}(x)(\phi^h)_{x_1}(x)dx$$
$$= \int_{-\infty}^{\infty}\int_{\boldsymbol{R}^2} \phi_{x_2}(x)(\phi^h)_{x_2}(x)dx.$$

Notice that we can change the order of integration by the estimate (2.9). Firstly, we compute the left term of (2.25):

(2.26) $$\int_{-\infty}^{\infty}\int_{\boldsymbol{R}^2} \phi_{x_1}(x)(\phi^h)_{x_1}(x)dx\,dh$$
$$= \int_{\boldsymbol{R}^2} \phi_{x_1}(x)\int_{-\infty}^{\infty} \phi_{x_1}(x_1-h, x_2)dh\,dx$$
$$= \int_{\boldsymbol{R}^2} \phi_{x_1}(x)\Big[\phi(h, x_2)\Big]_{-\infty}^{\infty} dx$$
$$= 0.$$

The equality is obvious by the estimate (2.8). Secondly, we compute the right term of (2.25):

(2.27) $$\int_{-\infty}^{\infty}\int_{\boldsymbol{R}^2} \phi_{x_2}(x)(\phi^h)_{x_2}(x)dx\,dh$$
$$= \int_{\boldsymbol{R}^2} \phi_{x_2}(x)\int_{-\infty}^{\infty} \phi_{x_2}(x_1-h, x_2)dh\,dx$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{\infty}^{\infty} \phi_{x_2}(x_1, x_2)\, dx_1 \right\}^2 dx_2$$
$$> 0.$$

The positivity of integrals is shown by computing the form of ϕ_{x_2}. Notice that ϕ_{x_2} is continuous, so it is sufficient to check that $\phi_{x_2}(x_1, 0)$ is not zero. By (2.13), we can write ϕ_{x_2} as

$$\phi_{x_2}(x) = \frac{1}{\pi} \int_{B_+(0,R)} \omega(y) \frac{y_2\{(x_2 - y_2)(x_2 + y_2) - (x_1 - y_1)^2\}}{|x - y|^2 |x - \bar{y}|^2} dy \tag{2.28}$$

where ω is a function in T. Putting $x_2 = 0$ in (2.28) leads

$$\phi_{x_2}(x_1, 0) = -\frac{1}{\pi} \int_{B_+(0,R)} \omega(y) \frac{y_2}{(x_1 - y_1)^2 + y_2^2} dy. \tag{2.29}$$

Since ω and the integral kernel are positive in $B_+(0, R)$, $\phi_{x_2}(x_1, 0)$ is always nonzero. The results of computation (2.26) and (2.27) leads a contradiction, and we get a conclusion of the lemma. □

Proof of Theorem 1.3. Let ϕ the function in S that satisfies (2.21). Let $\phi^{\varepsilon}(x) = \phi(x/\varepsilon)$. Then

$$\{(\phi^{\varepsilon})_{x_1}(x)\}^2 - \{(\phi^{\varepsilon})_{x_2}(x)\}^2 = \frac{1}{\varepsilon^2} \{(\phi_{x_1})^2 - (\phi_{x_2})^2\}\left(\frac{x}{\varepsilon}\right)$$

and Lemma 2.1 is appliable. (Notice that $\phi_{x_1}^2 - \phi_{x_2}^2$ is in $L^1(\boldsymbol{R}^2)$: the estimate (2.9) leads this fact.) □

References

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