Algorithms for b-Functions, Induced Systems, and Algebraic Local Cohomology of D-Modules

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1. Introduction. Let K be an algebraically closed field of characteristic zero and let X be a Zariski open set of K^n with a positive integer n. We fix a coordinate system $x = (x_1, \dots, x_n)$ of X and write $\partial = (\partial_1, \dots, \partial_n)$ with $\partial_i := \partial/\partial x_i$. We denote by $\mathcal{D}_X$ the sheaf of algebraic differential operators on X (cf. [2], [3]).

We assume that (a presentation of) a coherent left $\mathcal{D}_X$-module $\mathcal{M}$ is given. Let u be a section of $\mathcal{M}$ and let $f = f(x)$ be an arbitrary polynomial of n variables. Let s be an indeterminate. If $\mathcal{M}$ is holonomic, then for each point p of $Y := \{x \in X \mid f(x) = 0\}$, there exist a germ $P(x, \partial, s)$ of $\mathcal{D}_X[s]$ at p and a polynomial $b(s) \in K[s]$ of one variable so that

(1.1) $$P(x, \partial, s)(f^{s+1}u) = b(s) f^s u$$

holds (cf. [11]). More precisely, (1.1) means that there exists a nonnegative integer m so that

$$Q := f^{m-s}(b(s) - P(x, \partial, s) f) f^s \in \mathcal{D}_X[s]$$

satisfies $Qu = 0$ in $\mathcal{M}[s] := K[s] \otimes_K \mathcal{M}$. A monic polynomial $b(s)$ of the least degree that satisfies (1.1) is called the (generalized) b-function for f and u. When $\mathcal{M}$ coincides with the sheaf $\mathcal{O}_X$ of regular functions and $u = 1$, we get the classical b-function (or the Bernstein-Sato polynomial) of f. Algorithms for computing the Bernstein-Sato polynomial have been given by several authors ([21], [25], [4], [16]) but not for an arbitrary f.

One of the main purposes of the present paper is to give algorithms for computing the b-function for u and f and for computing the algebraic local cohomology groups $\mathcal{H}^j_{[Y]}(\mathcal{M})$ $(j = 0,1)$ as left $\mathcal{D}_X$-modules (cf. [11] for the definition). The algorithm for the local cohomology groups needs some information on the b-function.

These algorithms are actually obtained as byproducts of the solution of more general problems as follows:

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_{K \times X}$-module. For the sake of simplicity, let us assume here that a section u of $\mathcal{M}$ generates $\mathcal{M}$. We identify X with the subset $\{(t, x) \in K \times X \mid t = 0\}$ of $K \times X$. Then the b-function of u along X at $p \in X$ is a nonzero polynomial $b(s) \in K[s]$ of the least degree that satisfies

$$(b(t\partial_t) + tP(t, x, t\partial_t, \partial))u = 0$$

with a germ $P(t, x, t\partial_t, \partial)$ of $\mathcal{D}_{K \times X}$ at p, where we write $\partial_t := \partial/\partial t$. $\mathcal{M}$ is called *specializable* along X at p if such $b(s)$ exists.

We first present an algorithm which computes $b(s)$, or determines that there is none, by using a kind of Gröbner basis for the Weyl algebra related to a filtration introduced by Kashiwara [12]. Such Gröbner bases were used in [18], [19], [20].

If $\mathcal{M}$ is specializable, then its induced system to X is the complex of left $\mathcal{D}_X$-modules $\mathcal{M}^{\bullet}_X$ whose cohomology groups are coherent $\mathcal{D}_X$-modules. We also obtain an algorithm of computing the cohomology groups of $\mathcal{M}^{\bullet}_X$ by using an FW-Gröbner basis. These algorithms for the b-function and the induced system, combined with a viewpoint of Malgrange [17], provide algorithms for the b-function for a polynomial (and a section of a holonomic system), and for the algebraic local cohomology groups.

When K coincides with the field $\boldsymbol{C}$ of complex numbers, we can consider the problems explained so far with $\mathcal{D}_X$ replaced by the sheaf $\mathcal{D}_X^{\mathrm{an}}$ of *analytic* differential operators. Then our algorithms yield correct solutions also in this analytic case if the left $\mathcal{D}_X^{\mathrm{an}}$-module $\mathcal{M}^{\mathrm{an}}$ in question is written in the form $\mathcal{M}^{\mathrm{an}} = \mathcal{D}_X^{\mathrm{an}} \otimes_{\mathcal{D}_X} \mathcal{M}$ with a coherent $\mathcal{D}_X$-module $\mathcal{M}$ whose presentation is given explicitly.

We have implemented the algorithms by using a computer algebra system Kan [24]. Details of the present paper will appear elsewhere.

2. Gröbner bases. Let us denote by A_n and by A_{n+1} the Weyl algebra on n variables x, and the Weyl algebra on $n+1$ variables (t, x) respectively with coefficients in K. Let r be a positive integer and put $L := N^{2+2n} = N \times N \times N^n \times N^n$ with $N := \{0,1,2,\dots\}$. An element P of $(A_{n+1})^r$ is written in a finite sum

$$(2.1) \qquad P = \sum_{i=1}^{r} \sum_{(\mu,\nu,\alpha,\beta)\in L} a_{\mu\nu\alpha\beta i} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial^{\beta} e_i$$

with $a_{\mu\nu\alpha\beta i} \in K$, $e_1 := (1,0,\dots,0),\dots, e_r := (0,\dots,0,1)$, $x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^{\beta} := \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ for $\alpha = (\alpha_1,\dots,\alpha_n)$, $\beta = (\beta_1,\dots,\beta_n) \in N^n$.

For each integer m, we set

$$F_m((A_{n+1})^r) := \{P = \sum_{i=1}^{r} \sum_{(\mu,\nu,\alpha,\beta)\in L} a_{\mu\nu\alpha\beta i} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial^{\beta} e_i \mid a_{\mu\nu\alpha\beta i} = 0 \text{ if } \nu - \mu > m\}.$$

Then $\{F_m((A_{n+1})^r)\}_{m\in Z}$ constitutes a filtration of $(A_{n+1})^r$. For a nonzero element P of $(A_{n+1})^r$, the *F-order* $\mathrm{ord}_F(P)$ of P is defined as the least $m \in Z$ such that $P \in F_m((A_{n+1})^r)$.

Let $\prec_F$ be a total order on $L \times \{1,\dots,r\}$ which satisfies

(A-1) $(\alpha, i) \prec_F (\beta, j)$ implies $(\alpha + \gamma, i) \prec_F (\beta + \gamma, j)$ for any $\alpha, \beta, \gamma \in L$ and $i, j \in \{1,\dots,r\}$;

(A-2) if $\nu - \mu < \nu' - \mu'$, then $(\mu, \nu, \alpha, \beta, i) \prec_F (\mu', \nu', \alpha', \beta', j)$ for any $\alpha, \beta, \alpha', \beta' \in N^n$, $\mu, \nu, \mu', \nu' \in N$ and any $i, j \in \{1,\dots,r\}$;

(A-3) $(\mu, \mu, \alpha, \beta, i) \succeq_F (0,0,0,0,i)$ for any $\mu \in N$, $\alpha, \beta \in N^n$, $i \in \{1,\dots,r\}$.

Let P be a nonzero element of $(A_{n+1})^r$ which is written in the form (2.1). Then the *leading exponent* $\mathrm{lexp}_F(P) \in L \times \{1,\dots,r\}$ of P with respect to $\prec_F$ is defined as the maximum element

$$\max\{(\mu, \nu, \alpha, \beta, i) \mid a_{\mu\nu\alpha\beta i} \neq 0\}$$

with respect to the order $\prec_F$. The set of leading exponents $E_F(N)$ of a subset N of $(A_{n+1})^r$ is defined by

$$E_F(N) := \{\mathrm{lexp}_F(P) \mid P \in N \setminus \{0\}\}.$$

Definition 2.1. A finite set G of generators of a left A_{n+1}-submodule N of $(A_{n+1})^r$ is called an FW-*Gröbner basis* of N if we have

$$E_F(N) = \bigcup_{P\in G} (\mathrm{lexp}_F(P) + L),$$

where we write

$$(\alpha, i) + L = \{(\alpha + \beta, i) \mid \beta \in L\}$$

for $\alpha \in L$ and $i \in \{1,\dots,r\}$.

Since the order $\prec_F$ is not a well-order, the Buchberger algorithm ([5], [9], [6], [22]) for computing Gröbner bases does not work directly. In order to bypass this difficulty to obtain an algorithm of computing FW-Gröbner bases, we use the homogenization technique.

Definition 2.2. For $\lambda, \mu, \nu, \lambda', \mu', \nu' \in N$ and $\alpha, \beta, \alpha', \beta' \in N^n$, an order $\prec_H$ on $L_1 \times \{1,\dots,r\}$ with $L_1 := N \times L$ is defined so that we have $(\lambda, \mu, \nu, \alpha, \beta, i) \prec_H (\lambda', \mu', \nu', \alpha', \beta', j)$ if and only if one of the following conditions holds:

(1) $\lambda < \lambda'$;

(2) $\lambda = \lambda'$, $(\mu + l, \nu, \alpha, \beta, i) \prec_F (\mu' + l', \nu', \alpha', \beta', j)$ with $l, l' \in N$ such that $\nu - \mu - l = \nu' - \mu' - l'$;

(3) $(\lambda, \nu, \alpha, \beta, i) = (\lambda', \nu', \alpha', \beta', j)$, $\mu < \mu'$

This definition is independent of the choice of l, l' in view of the condition (A-1).

Lemma 2.3. (1) *$\prec_H$ is a well-order.*

(2) *If $\nu - \mu - \lambda = \nu' - \mu' - \lambda'$, then $(\lambda, \mu, \nu, \alpha, \beta, i) \prec_H (\lambda', \mu', \nu', \alpha', \beta', j)$ if and only if $(\mu, \nu, \alpha, \beta, i) \prec_F (\mu', \nu', \alpha', \beta', j)$.*

Definition 2.4. An element P of $(A_{n+1}[x_0])^r$ of the form

$$P = \sum_{i=1}^{r} \sum_{\lambda,\mu,\nu,\alpha,\beta} a_{\lambda\mu\nu\alpha\beta i} x_0^{\lambda} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial^{\beta} e_i$$

is said to be *F-homogeneous* of order m if $a_{\lambda\mu\nu\alpha\beta i} = 0$ whenever $\nu - \mu - \lambda \neq m$.

Definition 2.5. For an element P of $(A_{n+1})^r$ of the form (2.1), put $m := \min\{\nu - \mu \mid a_{\mu\nu\alpha\beta i} \neq 0$ for some $\alpha, \beta \in N^n$ and $i \in \{1,\dots,r\}\}$. Then the *F-homogenization* $P^h \in (A_{n+1}[x_0])^r$ of P is defined by

$$P^h := \sum_{i=1}^{r} \sum_{\mu,\nu,\alpha,\beta} a_{\mu\nu\alpha\beta i} x_0^{\nu-\mu-m} t^{\mu} x^{\alpha} \partial_t^{\nu} \partial^{\beta} e_i.$$

P^h is F-homogeneous of order m.

Proposition 2.6. *Let N be a left $A_{n+1}[x_0]$-submodule of $(A_{n+1}[x_0])^r$ generated by F-homogeneous operators. Then there exists an H-Gröbner basis (i.e. a Gröbner basis with respect to $\prec_H$) consisting of F-homogeneous operators. Moreover, such an H-Gröbner basis can be computed by the Buchberger algorithm.*

Theorem 2.7. *Let N be a left A_{n+1}-submodule of $(A_{n+1})^r$ generated by $P_1, \dots, P_d \in (A_{n+1})^r$. Let us denote by N^h the left $A_{n+1}[x_0]$-submodule of $(A_{n+1}[x_0])^r$ generated by $(P_1)^h, \dots, (P_d)^h$. Let $G = \{Q_1(x_0), \dots, Q_k(x_0)\}$ be an H-Gröbner basis of N^h consisting of F-homogeneous operators. Then $G(1) := \{Q_1(1), \dots, Q_k(1)\}$ is an FW-Gröbner basis of N.*

Let us denote by $\mathscr{D}_{K\times X}|_X$ the sheaf theoretic restriction of $\mathscr{D}_{K\times X}$ to $X = X \times \{0\}$. Then for a germ Q of $(\mathscr{D}_{K\times X}|_X)^r$ at $p \in X$, there exist $P \in$

$(A_{n+1})^r$ and $a(t, x) \in K[t, x]$ with $a(0, p) \neq 0$ so that $Q = a(t, x)^{-1}P$. For each integer m, we put

$$F_m((\mathscr{D}_{K\times X}|_X)^r)_p := \{a^{-1}P \mid P \in F_m((A_{n+1})^r),\ a = a(t, x) \in K[t, x], a(0, p) \neq 0\}.$$

For a germ Q of $(\mathscr{D}_{K\times X}|_X)^r$ at p, its F-order $\mathrm{ord}_F(Q)$ is defined as the minimum $m \in \boldsymbol{Z}$ so that $P \in F_m((\mathscr{D}_{K\times X}|_X)^r)$. Put $m := \mathrm{ord}_F(Q)$ and let $a(t, x) \in K[t, x]$ and $P \in (A_{n+1})^r$ be as above. Suppose that P is written in the form (2.1). Then the *formal symbol* $\hat{\sigma}(Q)$ of Q is defined by

$$\hat{\sigma}(Q) = \hat{\sigma}_m(Q) := a(0, x)^{-1} \sum_{i=1}^{r} \sum_{\nu-\mu=m} a_{\mu\nu\alpha\beta i} t^\mu x^\alpha \partial_t^\nu \partial^\beta e_i.$$

Definition 2.8. Let P be a nonzero element of $(A_{n+1})^r$ (resp. $(\mathscr{D}_{K\times X}|_X)^r$) of F-order m. Then we define $\psi(P)(s) \in (A_n[s])^r$ (resp. $(\mathscr{D}_X[s])^r$), by

$$\hat{\sigma}_0(t^m P) = \psi(P)(t\partial_t) \text{ if } m \geq 0,$$
$$\hat{\sigma}_0(\partial_t^{-m} P) = \psi(P)(t\partial_t) \text{ if } m < 0.$$

Theorem 2.9. *We use the same notation as in Theorem 2.7. Let $\mathscr{N}$ be the left $\mathscr{D}_{K\times X}|_X$-submodule of $(\mathscr{D}_{K\times X}|_X)^r$ generated by N. Let $\psi(\mathscr{N})$ be the left $\mathscr{D}_X[s]$-submodule of $(\mathscr{D}_X[s])^r$ generated by the set $\{\psi(P)(s) \mid P \in \mathscr{N}, \mathrm{ord}_F(P) = 0\}$. Then $\psi(\mathscr{N})$ is generated by $\psi(Q_1(1)), \ldots, \psi(Q_k(1))$.*

3. b-function of a D-module. Let $\mathscr{M}$ be a left coherent $\mathscr{D}_{K\times X}|_X$-module on X. We assume that a left A_{n+1}-submodule N of $(A_{n+1})^r$ is given explicitly so that $\mathscr{M} = \mathscr{D}_{K\times X}|_X \otimes_{A_{n+1}} M$ holds with $M := (A_{n+1})^r/N$. Set $\mathscr{N} := \mathscr{D}_{K\times X}|_X \otimes_{A_{n+1}} N \subset (\mathscr{D}_{K\times X}|_X)^r$. For each integer m, put

$$F_m(\mathscr{N}) := \mathscr{N} \cap F_m((\mathscr{D}_{K\times X}|_X)^r),$$
$$F_m(\mathscr{M}) := F_m((\mathscr{D}_{K\times X}|_X)^r)/F_m(\mathscr{N}).$$

Then $\{F_m(\mathscr{M})\}_{m\in\boldsymbol{Z}}$ is a filtration of $\mathscr{M}$ satisfying

$$F_k(\mathscr{D}_{K\times X}|_X)F_m(\mathscr{M}) = F_{k+m}(\mathscr{M})$$

for any $k, m \in \boldsymbol{Z}$. The b-function $b(s, p) \in K[s]$ of $\mathscr{M}$ (with respect to the filtration $\{F_m(\mathscr{M})\}$) at $p \in X$ is the monic polynomial $b(s, p)$ of s of the least degree, if any, that satisfies

$$(3.1) \qquad b(t\partial_t, p)(F_0(\mathscr{M})/F_{-1}(\mathscr{M}))_p = 0.$$

If such $b(s, p)$ exists, $\mathscr{M}$ is called specializable along X at p. It is known that if $\mathscr{M}$ is holonomic, then $\mathscr{M}$ is specializable at any $p \in X$ ([13], [14]).

Let $\boldsymbol{G}$ be an FW-Gröbner basis of N, which can be computed by the homogenization and the Buchberger algorithm with a set of generators as input (Theorem 2.7). Put $\psi(\boldsymbol{G}) := \{\psi(P) \mid P \in \boldsymbol{G}\}$ and let $\psi(N)$ be the left $A_n[s]$-submodule of $(A_n[s])^r$ generated by $\psi(\boldsymbol{G})$. Let $\prec_D$ be a total order on $L_0 \times \{1, \ldots, r\}$ with $L_0 := \boldsymbol{N}^{1+2n}$ which satisfies (A-1) with L replaced by L_0 and

(A-4) $(\alpha, i) \succ_D (0, i)$ for any $\alpha \in L_0 \setminus \{0\}$ and $i \in \{1, \ldots, r\}$;

(A-5) $|\beta| < |\beta'|$ implies $(\mu, \alpha, \beta, i) \prec_D (\mu', \alpha', \beta', j)$ for any $\mu, \mu' \in \boldsymbol{N}$, $\alpha, \alpha', \beta, \beta' \in \boldsymbol{N}^n$, $i, j \in \{1, \ldots, r\}$.

Theorem 3.1. *Under the above assumptions, let $\boldsymbol{G}_1$ be a Gröbner basis of $\psi(N)$ with respect to $\prec_D$ and put $\boldsymbol{G}_0 := \boldsymbol{G}_1 \cap K[s, x]^r$. Let $\mathscr{T}$ be the $\mathscr{O}_X[s]$-submodule of $(\mathscr{O}_X[s])^r$ generated by $\boldsymbol{G}_0$. Then $\mathscr{M}$ is specializable at p if and only if $\mathscr{T}_p \cap K[s]e_i \neq \{0\}$. If $\mathscr{M}$ is specializable, then its b-function $b(s, p)$ is the monic polynomial of s of the least degree that satisfies $b(s, p)e_i \in \mathscr{T}_p \cap K[s]^r$ for any $i = 1, \ldots, r$.*

Since we have a set of generators of $\mathscr{T}$, it is easy to compute $\mathscr{T} \cap K[s]^r$. This can be done, e.g., by primary decomposition of the $K[s, x]$-submodule of $K[s, x]^r$ which is generated by $\boldsymbol{G}_0$ (cf.[8]). Thus we obtain an algorithm of determining if $\mathscr{M}$ is specializable at each point of X and of computing the b-function if that is the case.

4. Induced system. We retain the notation of the preceding section. The induced system of $\mathscr{M}$ to X is the complex

$$\mathscr{M}_X^\cdot : 0 \to \mathscr{M} \xrightarrow{t} \mathscr{M} \to 0$$

of left $\mathscr{D}_X$-modules, where the homomorphism t denotes the one defined by $t(u) = tu$ for $u \in \mathscr{M}$. Let us write $\mathscr{M}_X := \mathscr{M}/t\mathscr{M}$. For each integer m, we put

$$\mathrm{gr}_m^F(\mathscr{M}) := F_m(\mathscr{M})/F_{m-1}(\mathscr{M}).$$

Lemma 4.1. *Assume that $b(s) \in K[s]$ satisfies $b(t\partial_t)\mathrm{gr}_0^F(\mathscr{M}) = 0$. Then the homomorphism $t : \mathrm{gr}_{k+1}^F(\mathscr{M}) \to \mathrm{gr}_k^F(\mathscr{M})$ is bijective if $b(k) \neq 0$.*

Proposition 4.2. *Assume that $b(s) \in K[s]$ satisfies $b(t\partial_t)\mathrm{gr}_0^F(\mathscr{M}) = 0$. Put*

$$k_1 := \max\{k \in \boldsymbol{Z} \mid b(k) = 0\},$$
$$k_0 := \min\{k \in \boldsymbol{Z} \mid b(k) = 0\}.$$

Then $\mathscr{M}_X^\cdot$ is quasi-isomorphic to the complex

$$0 \to F_{k_1+1}(\mathscr{M})/F_{k_0}(\mathscr{M}) \xrightarrow{t} F_{k_1}(\mathscr{M})/F_{k_0-1}(\mathscr{M}) \to 0$$

of left $\mathscr{D}_X$-modules. In particular, $t : \mathscr{M} \to \mathscr{M}$ is bijective if $b(k) \neq 0$ for any $k \in \boldsymbol{Z}$.

Proposition 4.3. *Assume that there exists $b(s) \in K[s]$ and $m \in \boldsymbol{N}$ so that*

$$b(t\partial_t)\partial_t^m \mathrm{gr}_0^F(\mathscr{M}) = 0.$$

Assume, moreover, $b(k) \neq 0$ for any $k \in \boldsymbol{Z}$. Then the homomorphism $t : \mathscr{M} \to \mathscr{M}$ is injective.

Let P be an element of $F_m((A_{n+1})^r)$. Then we can write P in the form

$$P = \sum_{i=1}^{r}\sum_{k=0}^{m} P_{ik}(t\partial_t, x, \partial)\partial_t{}^k e_i + R$$

uniquely with $P_{ik} \in A_n[t\partial_t]$ and $R \in F_{-1}((A_{n+1})^r)$. Then we put

$$\rho(P, k_0) := \sum_{i=1}^{r}\sum_{k=k_0}^{m} P_{ik}(0, x, \partial)\partial_t{}^k e_i$$

for each integer k_0 with $0 \le k_0 \le m$.

Theorem 4.4. *Assume that $b(s) \in K[s]$ satisfies $b(t\partial_t)\mathrm{gr}_0^F(\mathcal{M}) = 0$. Put*

$$k_1 := \max\{k \in \boldsymbol{Z} \mid b(k) = 0\},$$
$$k_0 := \max\{0, \min\{k \in \boldsymbol{Z} \mid b(k) = 0\}\}.$$

(We have $k_1 = m - 1$ and $k_0 = 0$ under the assumption of Proposition 4.3.) Let $\boldsymbol{G}$ be an FW-Gröbner basis of N. Then we have an isomorphism

$$\mathcal{M}_X \simeq \Big(\bigoplus_{i=1}^{r}\bigoplus_{k=k_0}^{k_1} \mathcal{D}_X\partial_t{}^k e_i\Big)/\mathcal{N}_X$$

of left $\mathcal{D}_X$-modules, where $\mathcal{N}_X$ is the left $\mathcal{D}_X$-module generated by a finite set

$$\{\rho(\partial_t^j P, k_0) \mid P \in \boldsymbol{G}, j \in \boldsymbol{N},$$
$$k_0 \le j + \mathrm{ord}_F(P) \le k_1\}.$$

Our next aim is to give an algorithm for computing the structure of the kernel $\mathcal{H}^{-1}(\mathcal{M}_X^{\cdot})$ of $t : \mathcal{M} \to \mathcal{M}$ as a left $\mathcal{D}_X$-module. For two integers $k_0 \le k_1$, put

$$\tilde{\mathcal{D}}^{(k_0,k_1)} := \bigoplus_{i=1}^{r}\bigoplus_{k=k_0}^{k_1} \mathcal{D}_X[t\partial_t]S_k e_i,$$

where we put $S_k := \partial_t{}^k$ if $k \ge 0$, and $S_k := t^{-k}$ if $k < 0$. Let P be a section of $(\mathcal{D}_{K\times X}|_X)^r$ of F-order m. Then we can write P uniquely in the form

$$P = \sum_{i=1}^{r}\sum_{k=-\infty}^{m} P_{ik}(t\partial_t, x, \partial)S_k e_i$$

with $P_{ik} \in (\mathcal{D}_X[t\partial_t])^r$. Then we define

$$\tau(P, k_0) := \sum_{i=1}^{r}\sum_{k=k_0}^{m} P_{ik}(t\partial_t, x, \partial)S_k e_i.$$

Proposition 4.5. *Let $\boldsymbol{G}$ be an FW-Gröbner basis of N. Then, for any integers $k_0 \le k_1$, we have an isomorphism*

$$F_{k_1}(\mathcal{M})/F_{k_0-1}(\mathcal{M}) \simeq \tilde{\mathcal{D}}^{(k_0,k_1)}/\mathcal{G}^{(k_0,k_1)}$$

of left $\mathcal{D}_X[t\partial_t]$-modules, where $\mathcal{G}^{(k_0,k_1)}$ is a left $\mathcal{D}_X[t\partial_t]$-module generated by a finite set

$$\{\tau(S_j P, k_0) \mid P \in \boldsymbol{G}, j \in \boldsymbol{Z},$$
$$k_0 \le j + \mathrm{ord}_F(P) \le k_1\}.$$

Let $\varphi : \tilde{\mathcal{D}}^{(k_0+1,k_1+1)} \to \tilde{\mathcal{D}}^{(k_0,k_1)}$ be a left $\mathcal{D}_X$-module homomorphism defined by

$$\varphi\Big(\sum_{i=1}^{r}\sum_{k=k_0}^{k_1} P_{i,k+1}(t\partial_t, x, \partial)S_{k+1}e_i\Big)$$
$$= \sum_{i=1}^{r}\sum_{k=k_0}^{k_1} P_{i,k+1}(t\partial_t - 1, x, \partial)T_k e_i$$

with

$$T_k := \begin{cases} S_k & (k \le -1) \\ t\partial_t S_k & (k \ge 0). \end{cases}$$

Theorem 4.6. *Under the same assumptions as in Proposition 4.2, we have an isomorphism*

$$\mathcal{H}^{-1}(\mathcal{M}_X^{\cdot}) \simeq \varphi^{-1}(\mathcal{G}^{(k_0,k_1)})/\mathcal{G}^{(k_0+1,k_1+1)}$$

as left $\mathcal{D}_X[t\partial_t]$-modules. Moreover, $\varphi^{-1}(\mathcal{G}^{(k_0,k_1)}) / \mathcal{G}^{(k_0+1,k_1+1)}$ is finitely generated as left $\mathcal{D}_X$-module.

The left $\mathcal{D}_X[t\partial_t]$-module $\varphi^{-1}(\mathcal{G}^{(k_0,k_1)})$ can be easily computed by the same method as for computing ideal intersection and quotient in the polynomial ring by means of Gröbner basis (cf. [7]). Then by eliminating $t\partial_t$ also by means of a Gröbner basis, we get an algorithm of computing a presentation of $\mathcal{H}^{-1}(\mathcal{M}_X^{\cdot})$ as a left $\mathcal{D}_X$-module.

5. Algebraic local cohomology. Let N be a left A_n-submodule of $(A_n)^r$ and put $M := (A_n)^r/N$ and $\mathcal{M} := \mathcal{D}_X \otimes_{A_n} M$. Let $f = f(x) \in K[x]$ be a polynomial and put $Y := \{x \in X \mid f(x) = 0\}$. Then the algebraic local cohomology group $\mathcal{H}^j_{[Y]}(\mathcal{M})$ has a structure of left $\mathcal{D}_X$-module and vanishes for $j \ne 0,1$ ([11]). Our purpose is to give an algorithm of computing $\mathcal{H}^j_{[Y]}(\mathcal{M})$ as a left $\mathcal{D}_X$-module.

Let $\mathcal{I}$ be a left ideal of $\mathcal{D}_{K\times X}$ generated by operators $t - f(x)$, $\partial_1 + (\partial f/\partial x_1)\partial_t, \ldots, \partial_n + (\partial f/\partial x_n)\partial_t$, and put $\mathcal{L} := \mathcal{D}_{K\times X}/\mathcal{I}$. Then by a method similar to that used by Malgrange [17], we get the following.

Theorem 5.1. *We have isomorphisms*

$$\mathcal{H}^j((\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L})^{\cdot}_X) \simeq \mathcal{H}^{j+1}_{[Y]}(\mathcal{M})$$

of left $\mathcal{D}_X$-modules for $j = -1,0$.

Let p_1 and p_2 be the projections of $X \times K \times X$ to X and to $K \times X$ respectively and put

$$\Delta := \{(x, t, y) \in X \times K \times X \mid x = y\}.$$

Then we have by [11]

$$\mathcal{M} \overset{\boldsymbol{L}}{\otimes}_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_\Delta \overset{\boldsymbol{L}}{\otimes}_{\mathcal{O}_{X\times K\times X}} (\mathcal{M} \hat{\otimes} \mathcal{L})$$

with

$$\mathcal{M} \hat{\otimes} \mathcal{L} := \mathcal{D}_{X\times K\times X} \otimes_{p_1^{-1}\mathcal{D}_X \otimes p_2^{-1}\mathcal{D}_{K\times X}} (p_1{}^{-1}\mathcal{M} \otimes_K p_2{}^{-1}\mathcal{L}),$$

where $\overset{\boldsymbol{L}}{\otimes}$ denotes the left derived functor of $\otimes$ in the derived category. In other words, $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}$ coincides with the induced system of $\mathcal{M} \hat{\otimes} \mathcal{L}$ along Δ. It is easy to see that $\mathcal{M} \hat{\otimes} \mathcal{L}$ is specializable along Δ (in fact, Δ is non-characteristic for this module). Hence we can compute $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}$ by applying Theorem 4.4 repeatedly with $k_0 = k_1 =$

0. Combining this fact with Theorems 4.4, 4.6, 5.1, we obtain an algorithm of computing $\mathscr{H}^j_{[Y]}(\mathscr{M})$ for $j = 0,1$.

Theorem 5.2. *Assume $r = 1$ and let $u \in \mathscr{M}$ be the residue class of $1 \in \mathscr{D}_X$. Let $\tilde{b}(s)$ be the b-function of $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{L}$ along X in the sense of Section 3 and let $b(s)$ be the b-function for f and u defined by (1.1) both at a point p of Y. Then we have the following*:

(1) *$b(s)$ divides $\tilde{b}(-s-1)$;*

(2) *if the homomorphism $f : \mathscr{M} \to \mathscr{M}$ defined by $f(v) = fv$ for $v \in \mathscr{M}$ is injective at p, then we have $b(s) = \pm\tilde{b}(-s-1)$;*

(3) *the homomorphism $f : \mathscr{M} \to \mathscr{M}$ is injective if and only if*
$$\mathscr{H}^{-1}((\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{L})^{\cdot}_X) = 0.$$

Thus the algorithm for $\tilde{b}(s)$ provides an algorithm to compute the b-function for f and u in generic cases. Since $f : \mathscr{O}_X \to \mathscr{O}_X$ is injective, we have an algorithm to compute the Bernstein-Sato polynomial of an arbitrary f.

It is also possible (in generic cases) to compute $\mathscr{H}^j_{[Y]}(\mathscr{M})$ for algebraic set Y of codimension greater than one. For example, let $f_1(x)$, $f_2(x)$ be two polynomials and put
$$Y_i := \{x \in X \mid f_i(x) = 0\} \quad (i = 1,2),$$
$$Y := Y_1 \cap Y_2.$$
Assume that $\mathscr{H}^j_{[Y_1]}(\mathscr{M}) = 0$ for $j \neq j_0$. Then we can compute
$$\mathscr{H}^j_{[Y]}(\mathscr{M}) = \mathscr{H}^{j-j_0}_{[Y_2]}(\mathscr{H}^{j_0}_{[Y_1]}(\mathscr{M}))$$
explicitly. The following computation was carried out by using Kan ([24]).

Example 5.3. Put $X = K^3$, $f_1 := x^2 - y^3$, $f_2 := y^2 - z^3$, and $Y := \{(x, y, z) \in X \mid f_1(x, y, z) = f_2(x, y, z) = 0\}$. Then we have $\mathscr{H}^j_{[Y]}(\mathscr{O}_X) = 0$ for $j \neq 2$ and
$$\mathscr{H}^2_{[Y]}(\mathscr{O}_X) \simeq \mathscr{D}_X / \mathscr{I},$$
where $\mathscr{I}$ is the left ideal of $\mathscr{D}_X$ generated by f_1, f_2 and
$$9x\partial_x + 6y\partial_y + 4z\partial_z + 30,$$
$$9z^2y^2\partial_x + 6z^2x\partial_y + 4yx\partial_z.$$
Let u_j be the residue class of f_j^{-1} in $\mathscr{H}^1_{[Y_j]}(\mathscr{O}_X) = \mathscr{O}_X[f_j^{-1}]/\mathscr{O}_X$ with $Y_j := \{(x, y, z) \mid f_j(x, y, z) = 0\}$. Then the b-function for f_2 and u_1 at $0 = (0,0,0)$ is
$$(s+1)\left(s+\frac{1}{12}\right)\left(s+\frac{5}{12}\right)\left(s+\frac{7}{12}\right)$$
$$\left(s+\frac{5}{6}\right)\left(s+\frac{11}{12}\right)\left(s+\frac{7}{6}\right),$$
while the b-function for f_1 and u_2 at 0 is
$$(s+1)\left(s+\frac{7}{18}\right)\left(s+\frac{11}{18}\right)\left(s+\frac{13}{18}\right)$$
$$\left(s+\frac{5}{6}\right)\left(s+\frac{17}{18}\right)\left(s+\frac{19}{18}\right)\left(s+\frac{7}{6}\right)\left(s+\frac{23}{18}\right).$$

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