# McKay Correspondence and Hilbert Schemes*)

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**Introduction.** A particular case in the superstring theory where a finite group $G$ acts upon the target Calabi-Yau manifold $M$ in the theory seems to attract both physicists' and mathematician's attention from various viewpoints. In order to obtain a correct conjectural formula of the Euler number of a smooth resolution of the quotient space $M/G$, physicists were led to define the following *orbifold Euler characteristic* [2], [3]

$$\chi(M, G) = \frac{1}{|G|}\sum_{gh=hg}\chi(M^{\langle g,h\rangle}),$$

where the summation runs over all the pairs $g$, $h$ of commuting elements of $G$, and $M^{\langle g,h\rangle}$ denotes the subset of $M$ of all the points fixed by both of $g$ and $h$. Then a conjecture of Vafa [2], [3] can be stated in mathematical terms as follows.

**Vafa's formula-conjecture.** *If a complex manifold $M$ has trivial canonical bundle and if $M/G$ has a (nonsingular) resolution of singularities $\widetilde{M/G}$ with trivial canonical bundle, then we have $\chi(\widetilde{M/G}) = \chi(M, G)$.*

In the special case where $M = A^n$ an $n$-dimensional affine space, $\chi(M, G)$ turns out to be the number of conjugacy classes, or equivalently the number of equivalence classes of irreducible $G$-modules. If $n = 2$, then the formula is therefore a corollary to the classical McKay correspondence between the set of exceptional irreducible divisors and the set of equivalence classes of irreducible $G$-modules [13].

If $n = 3$, then the existence of the above resolution as well as Vafa's formulae is known by the efforts of mathematicians [14], [17], [12], [18], [7], [8], [9], [19]. Except in these cases Vafa's formula is known to be true only in a few cases [6], for instance the case where $G$ is a symmetry group $S_m$ of $m$ letters for $n = 2m$ an arbitrary even integer [5] [15]. In this case $M/G = \mathrm{Symm}^m(A^2)$ and $\widetilde{M/G} = \mathrm{Hilb}^m(A^2)$ as we will see soon. A generalization of the classical McKay correspondence to an arbitrary $n$

is also known as an Ito-Reid (bijective) correspondence between the set of irreducible exceptional divisors in $\widetilde{M/G}$ and the set of certain conjugacy classes called junior ones [11].

In the present article we will report an interesting return-path from the case where $S_n$ acts on $A^{2n}$ to the two dimensional case with a different $G$. The analysis of the case leads us to a natural explanation for the classical McKay correspondence mentioned above. We will explain this more precisely in what follows.

Let $\mathrm{Symm}^n(A^2)$ ($\simeq \mathrm{Chow}^n(A^2)$) be the $n$-th symmetric product of $A^2$, that is by definition, the quotient of $n$-copies $A^{2n}$ of $A^2$ by the natural action of the symmetry group $S_n$ of $n$ letters. Let $\mathrm{Hilb}^n(A^2)$ be the Hilbert scheme of $A^2$ parametrizing all the 0-dimensional subschemes of length $n$. By [1] [4] $\mathrm{Hilb}^n(A^2)$ is a smooth resolution of $\mathrm{Symm}^n(A^2)$ with a holomorphic symplectic structure and trivial canonical bundle.

Let $G$ be an arbitrary finite subgroup of $SL(2, C)$. The group $G$ operates on $A^2$ so that it operates upon both $\mathrm{Hilb}^n(A^2)$ and $\mathrm{Symm}^n(A^2)$ canonically. Now we consider the particular case where $n$ is equal to the order of $G$. Then it is easy to see that the $G$-fixed point set $\mathrm{Symm}^n(A^2)^G$ in $\mathrm{Symm}^n(A^2)$ is isomorphic to the quotient space $A^2/G$. The $G$-fixed point set $\mathrm{Hilb}^n(A^2)^G$ in $\mathrm{Hilb}^n(A^2)$ is always nonsingular, but can be disconnected and not equidimensional. There is however a unique irreducible component of $\mathrm{Hilb}^n(A^2)^G$ dominating $\mathrm{Symm}^n(A^2)^G$, which we denote by $\mathrm{Hilb}^G(A^2)$. $\mathrm{Hilb}^G(A^2)$ is roughly speaking the Hilbert scheme parametrising all the $G$-orbits of length $|G|$. Since $\mathrm{Hilb}^G(A^2)$ inherits a holomorphic symplectic structure from

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$\mathrm{Hilb}^n(\boldsymbol{A}^2)$, $\mathrm{Hilb}^G(\boldsymbol{A}^2)$ is a smooth resolution of $\boldsymbol{A}^2/G$ with trivial canonical bundle (Theorem 1.3). The structure of $\mathrm{Hilb}^G(\boldsymbol{A}^2)$ is studied in detail by using the symmetric tensor representations of the group $G$.

Subsequently there emerges the classical McKay correspondence.

Let $\mathfrak{m}$ (resp. $\mathfrak{m}_S$) be the maximal ideal of the origin of $\boldsymbol{A}^2$ (resp. $\boldsymbol{A}^2/G$) and let $\mathfrak{n} = \mathfrak{m}_S \mathcal{O}_{\boldsymbol{A}}$. Any point $\mathfrak{p}$ of the exceptional set $E$ of $\mathrm{Hilb}^G(\boldsymbol{A}^2)$ is a $G$-invariant 0-dimensional subscheme $Z$ of $\boldsymbol{A}^2$ with support the origin, to which we associate a $G$-invariant ideal subsheaf $I$ of $\mathfrak{m}$ defining $Z$. Let $V(I) := I/\mathfrak{m}I + \mathfrak{n}$. The finite $G$-module $V(I)$ is isomorphic to a minimal $G$-submodule of $I$ generating the $\mathcal{O}_{\boldsymbol{A}^2}$-module $I$.

If $\mathfrak{p}$ is a smooth point of $E$, $V(I)$ is a nontrivial irreducible $G$-module. Meanwhile if $\mathfrak{p}$ is a singular point of $E$, then $V(I)$ is a sum of two mutually distinct nontrivial irreducible $G$-modules. For any nontrivial irreducible $G$-module $\rho$ we define a subset $E(\rho)$ of $E$ consisting of all $I \in \mathrm{Hilb}^G(\boldsymbol{A}^2)$ such that $V(I)$ contains $\rho$ as a $G$-submodule. We will see that $E(\rho)$ is a smooth rational curve. The map $\rho \mapsto E(\rho)$ gives a bijective correpondence (Theorem 3.1) between the set $\mathrm{Irr}(G)$ of all equivalence classes of nontrivial irreducible $G$-modules and the set $\mathrm{Irr}(E)$ of all irreducible components of $E$, which turns out to be the classical McKay correspondence [13].

## 1. The crepant (minimal) resolution.

**Lemma 1.1.** *Let $G$ be a finite group in $GL(2, \boldsymbol{C})$, $\mathrm{Hilb}^n(\boldsymbol{A}^2)^G$ the subset of $\mathrm{Hilb}^n(\boldsymbol{A}^2)$ consisting of all the points fixed by any element of $G$. Then $\mathrm{Hilb}^n(\boldsymbol{A}^2)^G$ is nonsingular.*

**Lemma 1.2.** *Let $G$ be a finite subgroup in $SL(2, \boldsymbol{C})$, $n$ the order of $G$ and $\mathrm{Symm}^n(\boldsymbol{A}^2)^G$ the subset of $\mathrm{Symm}^n(\boldsymbol{A}^2)$ consisting of all the points of $\mathrm{Symm}^n(\boldsymbol{A}^2)$ fixed by any element of $G$. Then $\mathrm{Symm}^n(\boldsymbol{A}^2)^G \simeq \boldsymbol{A}^2/G$.*

**Theorem 1.3.** *Let $G$ be a finite subgroup in $SL(2, \boldsymbol{C})$, $n$ the order of $G$. Then there is a unique irreducible component $\mathrm{Hilb}^G(\boldsymbol{A}^2)$ of $\mathrm{Hilb}^n(\boldsymbol{A}^2)^G$ dominating $\boldsymbol{A}^2/G$, which is a minimal resolution of $\boldsymbol{A}^2/G$ with trivial canonical line bundle.*

**Remark.** In what follows we identify a subscheme $Z$ and the ideal $I_Z$, so that we consider $I_Z \in \mathrm{Hilb}^n(\boldsymbol{A}^2)$.

## 2. $A_n$ case.

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\boldsymbol{A}^2}$ at the origin. Let $(x, y)$ be a system of coordinates of $\boldsymbol{A}^2$, $G$ a cyclic group of order $n+1$ and $\sigma$ a generator of $G$. Let $\varepsilon$ be a primitive $(n+1)$-th root of unity. We define the action of the generator $\sigma$ upon $\boldsymbol{C}^2$ by $(x, y) \mapsto (x, y) \cdot g = (\varepsilon x, \varepsilon^{-1} y)$. The simple singularity of type $A_n$ is the quotient of $\boldsymbol{A}^2$ by the cyclic group $G$.

**Lemma 2.1.** *$\mathrm{Hilb}^G(\boldsymbol{A}^2)$ is the union of the following $G$-invariant ideals of colength $n+1$;*

$$I(\Sigma) := \prod_{\mathfrak{p}\in\Sigma} \mathfrak{m}_{\mathfrak{p}} = (x^{n+1} - a^{n+1}, xy - ab, y^{n+1} - b^{n+1})$$

$$I_i(p_i : q_i) := (p_i x^i - q_i y^{n+1-i}, xy, x^{i+1}, y^{n+2-i})$$

*where $\Sigma$ is a $G$-orbit in $\boldsymbol{A}^2$ disjoint from the origin with $\#(\Sigma) = |G|$, $\mathfrak{p} := (a, b) \in \Sigma$, $\mathfrak{p} \neq (0,0)$, $1 \leq i \leq n$ and $[p_i, q_i] \in \mathbf{P}^1$.*

**Remark.** $\mathrm{Hilb}^G(\boldsymbol{A}^2)$ is the disjoint union of the subsets in Lemma 2. 1 except that $I_i(0:1) = I_{i+1}(1:0)$.

**Theorem 2.2.** *Let $a$ and $b$ be the parameters of $\boldsymbol{A}^2$ on which the group G acts by $g(a, b) = (\varepsilon a, \varepsilon^{-1} b)$. Let $S := \boldsymbol{A}^2/G$, $\tilde{S}$ the toric minimal resolution of $S$ and $U_i$ the affine charts of $\tilde{S}$ defined by*

$$\boldsymbol{A}^2/G \simeq \mathrm{Spec}\, \boldsymbol{C}[a^{n+1}, ab, b^{n+1}]$$

$$U_i := \mathrm{Spec}\, \boldsymbol{C}[s_i, t_i] \quad (1 \leq i \leq n+1)$$

*where we denote $s_i := a^i/b^{n+1-i}$, and $t_i := b^{n+2-i}/a^{i-1}$ under the usual notation of torus embeddings. Then the isomorphism of $\tilde{S}$ with $\mathrm{Hilb}^G(\boldsymbol{A}^2)$ is given by (the morphism defined by the universal property of $\mathrm{Hilb}^n(\boldsymbol{A}^2)$ from) the following two-dimensional flat families of $G$-invariant ideals of $\mathcal{O}_{\boldsymbol{A}^2}$ $(1 \leq i \leq n+1)$;*

$$\mathcal{I}_i(s_i, t_i) := (x^i - s_i y^{n+1-i}, xy - s_i t_i, y^{n+2-i} - t_i x^{i-1}).$$

## 3. Main theorem.

Let $G$ be a finite subgroup of $SL(2, \boldsymbol{C})$ and $\mathrm{Irr}(G)$ the set of all equivalence classes of nontrivial irreducible $G$-modules. Let $X = X_G := \mathrm{Hilb}^G(\boldsymbol{A}^2)$, $S = S_G := \boldsymbol{A}^2/G$, $\mathfrak{m}$ (resp. $\mathfrak{m}_S$) the maximal ideal of $X$ (resp. $S$) at the origin and $\mathfrak{n} := \mathfrak{m}_S \mathcal{O}_{\boldsymbol{A}^2}$. Let $\pi : X \to S$ be the natural morphism and $E$ the exceptional set of $\pi$. Let $\mathrm{Irr}(E)$ be the set of irreducible components of $E$. Any $I \in X$ contained in $E$ is a $G$-invariant ideal of $\mathcal{O}_{\boldsymbol{A}^2}$ which contains $\mathfrak{n}$. First we define

**Definition.** $V(I) := I/(\mathfrak{m}I + \mathfrak{n})$.

**Definition.** For any $\rho$, $\rho'$, and $\rho'' \in \mathrm{Irr}(G)$ we define

$$E(\rho) := \{I \in \mathrm{Hilb}^G(\boldsymbol{A}^2);\ V(I) \text{ contains a } G\text{-module } V(\rho)\}$$

$P(\rho, \rho') := \{I \in \mathrm{Hilb}^G(\boldsymbol{A}^2);\ V(I)$ contains a $G$-module $V(\rho) \oplus V(\rho')\}$
$Q(\rho, \rho', \rho'') := \{I \in \mathrm{Hilb}^G(\boldsymbol{A}^2);\ V(I)$ contains a $G$-module $V(\rho) \oplus V(\rho') \oplus V(\rho'')\}$.

**Definition.** Two irreducible $G$-modules $\rho$ and $\rho'$ are (McKay-) adjacent if $\rho \otimes \rho_{\mathrm{nat}} \supset \rho'$ or vice versa.

**Definition.** The McKay graph $\Gamma(\mathrm{Irr}(G))$ of $\mathrm{Irr}(G)$ is defined to be a graph whose vertices are $\mathrm{Irr}(G)$. Two vertices $\rho$ and $\rho'$ of $\Gamma(\mathrm{Irr}(G))$ are connected by a single edge if and only if $\rho$ and $\rho'$ are adjacent.

Then our main theorem is stated as follows.

**Theorem 3.1.** *Let $G$ be a finite subgroup of $SL(2, \boldsymbol{C})$. Then*

(1) *the map $\rho \mapsto E(\rho)$ is a bijective correspondence between $Irr(G)$ and $Irr(E)$,*

(2) *$E(\rho)$ is a smooth rational curve for any $\rho \in Irr(G)$,*

(3) *$P(\rho, \rho) = Q(\rho, \rho', \rho'') = \emptyset$ for any $\rho$, $\rho'\rho'' \in Irr(G)$.*

(4) *$P(\rho, \rho') \neq \emptyset$ if and only if $\rho$ and $\rho'$ are adjacent. In this case $P(\rho, \rho')$ is a (reduced) single point, where $E(\rho)$ and $E(\rho')$ intersect transversally.*

**Corollary 3.2.** *Let $Z^* := \mathrm{Hilb}^G(\boldsymbol{A}^2) \times_S \{0\}$ be a scheme-theoretic fiber of $\pi$ at the origin. Then $Z^*$ is a Cartier divisor of $X$ with $Z^* = \sum_{\rho \in \mathrm{Irr}(G)} (\deg \rho) E(\rho)$.*

Theorem 3. 1 is proved by describing all the ideals as we have done in section two for $A_n$. The details appear in [10] for $A_n$ and $D_n$ and in [16] for $E_6$, $E_7$ and $E_8$. By Theorem 3.1 $\Gamma(\mathrm{Irr}(G))$ is the same as $\Gamma(\mathrm{Irr}(E))$, the dual graph $\Gamma(\mathrm{Irr}(E))$ of $E$, in other words, the Dynkin diagram of $S_G$. We note that $\sum_{\rho \in \mathrm{Irr}(G)} (\deg \rho)\rho$ is the highest root in the root system on $\Gamma(\mathrm{Irr}(E))$ of $E$.

**Example.** With the notation in section two, we define characters $\rho_k$ of $G$ by $\rho_k(g) = \varepsilon^k (1 \leq k \leq n)$ or $(k \in \boldsymbol{Z}/(n+1)\boldsymbol{Z})$. Then we see that

$$V(I_k(p_k : q_k)) \simeq \begin{cases} \rho_1 & (k = 1, p_1 \neq 0) \\ \rho_1 + \rho_2 & (k, p_k) = (1,0), \text{ or } (k, q_k) = (2,0) \\ \rho_2 & (k = 2, p_2 q_2 \neq 0) \\ \rho_k + \rho_{k-1} & (q_k = 0, 2 \leq k \leq n) \\ \rho_k & (p_k q_k \neq 0) \\ \rho_k + \rho_{k+1} & (q_k = 0, 1 \leq k \leq n-1) \\ \rho_n & (k = n, q_n \neq 0 \end{cases}$$

It follows that $E(\rho_k) = \{I_k(p_k : q_k); [p_k : q_k] \in \boldsymbol{P}^1\}$ and $P(\rho_k, \rho_{k+1}) = \{I_k(0:1)\} = \{I_{k+1}(1:0)\}$. Since $\rho_k \otimes \rho_{\mathrm{nat}} = \rho_{k-1} + \rho_{k-1}$, we have $\Gamma(\mathrm{Irr}(G)) = \Gamma(\mathrm{Irr}(E))$.

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