Eigenvalues of the Laplacian Under Singular Variation of Domains—the Robin Problem with Obstacle of General Shape

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1. Introduction. Let M be a bounded domain in $\boldsymbol{R}^3$ with smooth boundary ∂M. Assume that $w = \{0\} \in M$. Let D be a domain with smooth boundary ∂D containing the origin $\{0\}$. Assume that $\boldsymbol{R}^3 \setminus D$ is connected. Let D_ε be the set given by $D_\varepsilon = \{x \in \boldsymbol{R}^3 ; \varepsilon^{-1}x \in D\}$. Let M_ε be the domain given by $M \setminus \overline{D_\varepsilon}$. Let $\mu_j(\varepsilon)$ be the j th eigenvalue of the Laplacian associated with the problem:

$$(1.1) \qquad \begin{aligned} -\Delta u(x) &= \lambda u(x) & x &\in M_\varepsilon \\ u(x) &= 0 & x &\in \partial M \\ ku(x) + (\partial/\partial\nu_x)u(x) &= 0 & x &\in \partial D_\varepsilon, \end{aligned}$$

where $k > 0$ is a constant and $\partial/\partial\nu_x$ denotes the derivative along the exterior normal direction with respect to ∂M. Let μ_j be the j th eigenvalue of the Laplacian associated with the following problem:

$$(1.2) \qquad \begin{aligned} -\Delta u(x) &= \lambda u(x) & x &\in M \\ u(x) &= 0 & x &\in \partial M. \end{aligned}$$

In this paper we give a sketch of the following

Theorem. *Fix j. Fix an arbitrary $\tau \in (0,1)$. Assume that μ_j is a simple eigenvalue. Then,*

$$\mu_j(\varepsilon) - \mu_j = k\,|\,\partial D\,|\,\varepsilon^2\varphi_j(w)^2 + O(\varepsilon^{2+\tau}).$$

Here $\varphi_j(x)$ is the L^2 normalized eigenfunction associated with μ_j. Here $|\,\partial D\,|$ is the surface area of ∂D.

Remark. See, for related topics to [5], Besson [1], Chavel and Feldman [2], Courtois [3], Roppongi [6].

2. Sketch of our proof of Theorem. Fix j. Let μ_j be a simple j th eigenvalue. Then, we can prove that $\mu_j(\varepsilon)$ is simple for any $0 < \varepsilon < \varepsilon_0$. Let $\varphi_j(\varepsilon)$ be L^2 normalized j th eigenfunction of $-\Delta$ associated with $\mu_j(\varepsilon)$. Let $d\sigma_x$ be two dimensional surface measure and ∇_t be a tangential gradient on the tangent space $T(\partial D_\varepsilon)$ at $x \in \partial D_\varepsilon$. Let H_1 be the first mean curvature with respect to inner normal vector at ∂D_ε. We have the following Hadamard's variational formula. See [4].

$$(2.1) \qquad \begin{aligned} &\mu_j'(\varepsilon) \\ &= \int_{\partial D_\varepsilon} (-\,|\,\nabla_t\varphi_j(\varepsilon)\,|^2 + \mu_j(\varepsilon)\varphi_j(\varepsilon)^2 \\ &\qquad + (k^2 + k(n-1)H_1)\varphi_j(\varepsilon)^2)(\nu_x \cdot n_x)\,d\sigma_x, \end{aligned}$$

where n_x is the unit vector along $\overrightarrow{wx}$ direction and $(\nu_x \cdot n_x)$ is the inner product.

To prove the Theorem we use the relation

$$\mu_j(\varepsilon) - \mu_j = \int_0^\varepsilon \mu_j'(s)\,ds$$

where μ_j is a simple eigenvalue.

We need to examine the properties of $\varphi_j(\varepsilon)$, $\nabla_t\varphi_j(\varepsilon)$ for small $\varepsilon > 0$ to obtain Theorem observing (2.1).

We can prove the following

Lemma 2.1. *Fix any positive number θ. Assume that μ_j is simple. Then,*

$$\max_{\overline{M}_\varepsilon} |\,\varphi_j(\varepsilon) - \varphi_j\,| = O(\varepsilon^{1-\theta})$$

is valid, if we take $\varphi_j(\varepsilon)$ such that

$$\int_{M_\varepsilon} \varphi_j(\varepsilon)(x)\varphi_j(x)\,dx > 0.$$

We also have the following

Lemma 2.2. *We have*

$$\int_{\partial D_\varepsilon} |\,\nabla_t\varphi_j(\varepsilon)(x)\,|^2 d\sigma_x = O(\varepsilon^2).$$

Then,

$$\begin{aligned} \mu_j'(\varepsilon) &= O(\varepsilon^2) + \int_{\partial D_\varepsilon} k(n-1)H_1\varphi_j(\varepsilon)^2(\nu_x \cdot n_x)\,d\sigma_x \\ &= O(\varepsilon^2) + \int_{\partial D_\varepsilon} k(n-1)H_1\varphi_j^2(\nu_x \cdot n_x)\,d\sigma_x \\ &\qquad + O(\varepsilon^2)O(\varepsilon^{-1})O(\varepsilon^{1-\theta}) \\ &= O(\varepsilon^{2-\theta}) + k(n-1) \\ &\qquad \left(\int_{\partial D_\varepsilon} H_1(\nu_x \cdot n_x)\,d\sigma_x\right)\varphi_j(w)^2 \end{aligned}$$

for any $\theta > 0$. Therefore,

$$\begin{aligned} \mu_j(\varepsilon) &= \mu_j + O(\varepsilon^{3-\theta}) + k\int_0^\varepsilon \left(\frac{d}{ds}\,|\,\partial D_s|\right)\varphi_j(w)^2 ds \\ &= \mu_j + k\,|\,\partial D\,|\,\varepsilon^2\varphi_j(w)^2 + O(\varepsilon^{3-\theta}). \end{aligned}$$

We can prove Theorem by using Lemma 2.1

and 2.2.

3. On Lemma 2.1. To prove Lemma 2.1 we need some steps. Let $G(x, y)$ be Green's fuction of $-\Delta$ associated with (1.2). Let $G_\varepsilon(x, y)$ be Green's function of $-\Delta$ which satisfy boundary conditions:

$$G_\varepsilon(x, y) = 0 \quad x \in \partial M,\ y \in M_\varepsilon$$

$$kG_\varepsilon(x, y) + (\partial/\partial\nu_x) G_\varepsilon(x, y) = 0, \quad x \in \partial D_\varepsilon,\ y \in M_\varepsilon.$$

We put

$$\boldsymbol{G}f(x) = \int_M G(x, y) f(y) dy$$

$$\boldsymbol{G}_\varepsilon f(x) = \int_{M_\varepsilon} G_\varepsilon(x, y) g(y) dy.$$

We have the following Lemma

Lemma 3.1. *We have*

$$\| \varphi_j(\varepsilon) \|_{L^\infty(M_\varepsilon)} = O(1).$$

Lemma 3.1 can be obtained by the relation $\varphi_j(\varepsilon) = \mu_j(\varepsilon) \boldsymbol{G}_\varepsilon \varphi_j(\varepsilon)$

Proof of Lemma 3.2. We put

$$u = (\boldsymbol{G}_\varepsilon - \boldsymbol{G}\chi) \varphi_j(\varepsilon).$$

Then,

$$\Delta u(x) = 0 \quad x \in M_\varepsilon$$

$$u(x) = 0 \quad x \in \partial M$$

and $ku(x) + (\partial/\partial\nu_x) u(x) = \beta(x) \quad x \in \partial D_\varepsilon$, where

$$\beta(x) = -kG\chi\varphi_j(\varepsilon)(x) - (\partial/\partial\nu_x) G\chi\varphi_j(x).$$

Here χ is the characteristic function of M_ε. Then, $\beta(x) = O(1)$. And we get Lemma 3.2 by the Green formula.

Lemma 3.3. *We have*

$$\| (\boldsymbol{G}_\varepsilon - \mu_j^{-1}) \chi \varphi_j \|_{L^2(M_\varepsilon)} = O(\varepsilon).$$

Proof of Lemma 2.1. We have the eigenfunction expansion

$$\boldsymbol{G}_\varepsilon f = \sum_{k=1}^{\infty} \mu_k(\varepsilon)^{-1} (\varphi_k(\varepsilon), f) \varphi_k(\varepsilon),$$

where $(,)$ is the inner product on $L^2(M_\varepsilon)$. Then,

$$\| (\boldsymbol{G}_\varepsilon - \mu_j^{-1}) \chi \varphi_j \|^2_{L^2(M_\varepsilon)} = O(\varepsilon^2)$$

implies

$$\sum_{k=1, k\neq j}^{\infty} (\varphi_k(\varepsilon), \chi\varphi_j)^2 = O(\varepsilon^2).$$

Therefore,

$$\| \chi\varphi_j - (\varphi_j(\varepsilon), \chi\varphi_j) \varphi_j(\varepsilon) \|_{L^2(M_\varepsilon)} = O(\varepsilon). \tag{3.1}$$

We know that

$$\int_{M_\varepsilon} \varphi_j(x)^2 dx = 1 + O(\varepsilon^3).$$

By taking a square of (3.1) we have

$$\| \chi\varphi_j \|^2_{L^2(M_\varepsilon)} - (\varphi_j(\varepsilon), \chi\varphi_j)^2 = O(\varepsilon^2).$$

Therefore,

$$(\varphi_j(\varepsilon), \chi\varphi_j)^2 = 1 + O(\varepsilon^2).$$

Then,

$$(\varphi_j(\varepsilon), \chi\varphi_j) = \operatorname{sgn}(\varphi_j(\varepsilon), \chi\varphi_j)(1 + O(\varepsilon^2)).$$

We have

$$\begin{aligned}\varphi_j(\varepsilon) = {} & (\mu_j(\varepsilon) - \mu_j) \boldsymbol{G}_\varepsilon \varphi_j(\varepsilon) \\ & + \mu_j(\boldsymbol{G}_\varepsilon - \boldsymbol{G}\chi) \varphi_j(\varepsilon) \\ & + \mu_j \boldsymbol{G}\chi(\varphi_j(\varepsilon) - \operatorname{sgn}(\varphi_j(\varepsilon), \chi\varphi_j)\chi\varphi_j) \\ & + \operatorname{sgn}(\varphi_j(\varepsilon), \chi\varphi_j)\mu_j \boldsymbol{G}\chi\varphi_j.\end{aligned}$$

Then, we can get Lemma 2.1.

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