Construction of Jacobi Forms from Certain Combinatorial Polynomials

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Introduction. The concept of Jacobi polynomials as a certain generalization of weight enumerators of certain codes was introduced by Ozeki [9]. In this paper we first interprete these Jacobi polynomials (defined by Ozeki) as the homogeneous polynomials of 4 variables which are invariant under the 4-dimensional action of the 2-dimentional finite unitary reflection group of order 192 (No. 9 in Shephard-Todd [12]). Then we determine the Molien series and the structure of the invariant ring. In Section 2, we define a new map from the space of homogeneous Jacobi polynomials (i.e., the invariant ring) to a space of Jacobi forms (in the sense of Eichler-Zagier [5]). This map, which we believe is very important for the study of Jacobi forms, is an extension of the well-known Broué-Enguehard map (and the extended version of it due to Ozeki [9], see also [4, Proposition 5.4]) from the space of the weight enumerators of binary self-dual doubly even codes to a space of modular forms. (Note that modular forms are Jacobi forms of index 0.) We conclude this paper by mentioning the outlines of further generalizations concerning this new map. The purpose of this paper is to announce the new results. The details will be published in forthcoming papers which are in preparation.

We remark that the papers [7], [3], [11] also deal with extensions of Broué-Enguehard theorem in various directions, in particular to obtain Siegel modular forms.

1. The space of Jacobi polynomials. Let $G = \langle \sigma_1, \sigma_2 \rangle$ with

$$(1) \qquad \sigma_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix}$$

and
$$\sigma_2 = \begin{bmatrix} \sqrt{-1} & & & \\ & 1 & & \\ & & \sqrt{-1} & \\ & & & 1 \end{bmatrix}$$

be the subgroup of $GL(4, \mathbf{C})$ generated by the two elements σ_1 and σ_2. Then $|G| = 192$ and G is the finite unitary group generated by reflections (u.g.g.r) referred to as No. 9 in Shephard-Todd [12]. Let $R = \mathbf{C}[x_1, y_1, x_2, y_2]$ and let R^G be the ring of invariants under the group $G = \langle \sigma_1, \sigma_2 \rangle$ with the natural action (1). Then we can interprete R^G as the space of Jacobi polynomials (in the sense of Ozeki [9]) for binary self-dual doubly even codes.

To be more precise, let V be a vector space of dimension n over the binary field $GF(2)$. V is equipped with the usual inner product $u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$ in $GF(2)$ for $u = (u_1, v_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. We also define

$$u * v = \#\{j \mid 1 \leq j \leq n,\ u_j = v_j = 1\}.$$

Note that $u * u$ is the Hamming weight $w(u)$ of u in V. Let $\mathscr{C}$ be a binary self-dual doubly even code in V, i.e., $\mathscr{C}$ is a vector subspace (over $GF(2)$) of V satisfying

$$\mathscr{C} = \mathscr{C}^{\perp} := \{y \in V \mid x \cdot y = 0, \forall x \in \mathscr{C}\}$$

and $4 \mid w(u)$ for $\forall u \in \mathscr{C}$. For a binary self-dual doubly even code $\mathscr{C}$ in V and for a vector v in V, Ozeki [9] defines the polynomial $J(\mathscr{C}, v \mid X, Z)$ in X and Z by

$$J(\mathscr{C}, v \mid X, Z) = \sum_{u \in \mathscr{C}} X^{u*u} Z^{u*v}.$$

For each of such polynomial $J(\mathscr{C}, v \mid X, Z)$, we can naturally associate a homogeneous polynomial $J(\mathscr{C}, v \mid x_1, y_1, x_2, y_2)$ of degree n in x_1, y_1, x_2, y_2 by

$$J(\mathscr{C}, v \mid x_1, y_1, x_2, y_2) = \sum_{u \in \mathscr{C}} x_1^{u*u-u*v} y_1^{n-v*v-(u*u-u*v)} x_2^{u*v} y_2^{v*v-u*v}.$$

Note that the correspondence between the homogeneous and inhomogeneous Jacobi polynomials is one to one, and this correspondence gives an analogy with the one between the

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homogeneous and inhomogeneous weight enumerators of codes.

It is proved that $J(\mathscr{C}, v \mid x_1, y_1, x_2, y_2)$ is invariant under the actions of σ_1 and σ_2 (on $R = \boldsymbol{C}[x_1, y_1, x_2, y_2]$) defined in (1). The invariance by σ_1 comes from the MacWilliams like identity proved in Ozeki [9] for inhomogeneous polynomials $J(\mathscr{C}, v \mid X, Z)$. Alternatively, this is also proved from the fact that in the association scheme which is the direct product of the Hamming association schemes $H(d_1, q)$ and $H(d_2, q)$ with $d = d_1 + d_2$ the MacWilliams transformation (the second eigenmatrix) Q of the scheme is the Kronecker product of Q_1 and Q_2 where $Q_i (i = 1,2)$ are the MacWilliams transforms of $H(d_i, q) (i = 1,2)$. On the other hand, the invariance under σ_2 comes from the fact that the code $\mathscr{C}$ is doubly even.

Our first main theorem is the following Gleason type theorem.

Theorem 1.1 *The ring R^G has the Molien series*

(2) $$\Phi_G(t) = \sum_{i \geq 0} (\dim R_i^G) t^i$$
$$= 1 + 10t^8 + 40t^{16} + 130t^{24} + 283t^{32} + 513t^{40} + 883t^{48} + 1372t^{56} + 1994t^{64} + 2836t^{72} + \cdots$$
$$= \frac{1 + 8t^8 + 21t^{16} + 58t^{24} + 47t^{32} + 35t^{40} + 21t^{48} + t^{56}}{(1 - t^8)^2 (1 - t^{24})^2},$$

where R_i^G is the homogeneous part of R^G of total degree i in x_1, y_1, x_2, and y_2. Moreover, the last expression gives a homogeneous system of parameters (h.s.o.p.) in the sense of Stanley [13].

Ozeki [9] also considered formal Jacobi polynomials, the space of which is identified with the space of relative invariants R_χ^G with respect to the group $G = \langle \sigma_1, \sigma_2 \rangle$ defined before and the linear character χ defined by $\chi(\sigma_1) = -1$ and $\chi(\sigma_2) = 1$. R_χ^G is also identified with the space of absolute invariants R^H with respect to the subgroup H (which is the kernel of the linear character χ mentioned above) of G of index 2. The group H is another finite u.g.g.r. (No. 8 in Shephard-Todd [12]).

Theorem 1.2. *The ring R^H has the Molien series*

(3) $$\Phi_H(t) = \sum_{i \geq 0} (\dim R_i^H) t^i$$
$$= 1 + 10t^8 + 20t^{12} + 40t^{16} + 75t^{20} + 130t^{24} + 179t^{28} + 283t^{32} + 383t^{36} + 513t^{40} + 678t^{44} + 883t^{48} + 1078t^{52} + 1372t^{56} + \cdots$$
$$= \frac{1 + 8t^8 + 18t^{12} + 21t^{16} + 19t^{20} + 21t^{24} + 7t^{28} + t^{32}}{(1 - t^8)^2 (1 - t^{12})^2}.$$

Moreover, the last expression gives a homogeneous system of parameters (h.s.o.p.) in the sense of Stanley [13].

2. An extension of Broué-Enguehard map. A Jacobi form $f(\tau, z)$ of weight k and index $m (k, m \in \mathrm{N})$ on a subgroup $\Gamma \subset \Gamma_1 = SL(2, \boldsymbol{Z})$ of finite index is a holomorphic function defined on $\mathscr{H} \times \boldsymbol{C}$, where $\mathscr{H}$ is the upper half plane and $\boldsymbol{C}$ is the complex number field, satisfying the following two transformation laws:

(*i*) $\phi \mid_{k,m} M = \phi \; (M \in \Gamma)$, where
$$\left(\phi \mid_{k,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)(\tau, z) := (c\tau + d)^{-k} e^{2\pi i m \left(\frac{-cz^2}{c\tau + d}\right)} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right),$$

(*ii*) $\phi \mid_m X = \phi \; (M \in Z^2)$, where
$$(\phi \mid_m [\lambda, \mu])(\tau, z) = e^{2\pi i m (\lambda^2 \tau + 2\lambda z)} \phi(\tau, z + \lambda\tau + \mu),$$
and the condition (iii) on Fourier development:

(*iii*) for each $M \in SL(2, \boldsymbol{Z})$, $\phi \mid_{k,m} M$ has a Fourier development of the form $\sum c(n, r) q^n \zeta^r$ $(q = e^{2\pi i \tau}, \zeta = e^{2\pi i z})$ with $c(n, r) = 0$ unless $n \geq r^2/4m$.

(See Eichler-Zagier [5, page 9]. If we do not specify the group Γ, then we usually understand that $\Gamma = \Gamma_1 = SL(2, \boldsymbol{Z})$.)

In [1] Broué-Enguehard defined a map from the weight enumerators of self-dual doubly even binary codes to a space of modular forms. Here the length n of a code satisfies $n \equiv 0$ (*mod* 8) and the weight k of the corresponding modular form satisfies $k = n/2 \equiv 0$ (*mod* 4). Ozeki [9] observed that the same map gives an algebra isomorphism between the space of formal weight enumerators (which is isomorphic to $\boldsymbol{C}[x_1, y_1]^H$ where H is mentioned as before) and the space $\boldsymbol{C}[E_4, E_6]$ of modular forms. We learned that this was also essentially observed in Ebeling [4, Proposition 5.4].

The main purpose of this section, as well as of this paper, is to extend Broué-Enguehard correspondence by defining a new map from the space of Jacobi polynomials $R^G = \boldsymbol{C}[x_1, y_1, x_2, y_2]^G$ into the space of Jacobi forms. In order to state our main results, we recall the definitions of Jacobi theta functions.

Let

$$(4)\quad \begin{cases} \theta_0(\tau, z) = \sum_{n\in \boldsymbol{Z}}(-1)^n e^{\pi i n^2\tau + 2n\pi i z}, \\ \theta_2(\tau, z) = \sum_{n\in \boldsymbol{Z}} e^{\pi i (n+1/2)^2\tau + (2n+1)\pi i z}, \text{ and} \\ \theta_3(\tau, z) = \sum_{n\in \boldsymbol{Z}}{}^{n} e^{\pi i n^2\tau + 2n\pi i z} \end{cases}$$

Then they satisfy the following well known transformation laws (cf. [8], [10], etc.).

$$(5)\quad \begin{cases} \theta_0(-1/\tau, z/\tau) = \sqrt{\dfrac{\tau}{i}}\, e^{\pi i z^2/\tau}\theta_2(\tau, z), \\ \theta_2(-1/\tau, z/\tau) = \sqrt{\dfrac{\tau}{i}}\, e^{\pi i z^2/\tau}\theta_0(\tau, z), \text{ and} \\ \theta_3(-1/\tau, z/\tau) = \sqrt{\dfrac{\tau}{i}}\, e^{\pi i z^2/\tau}\theta_3(\tau, z). \end{cases}$$

If we define $\varphi_i(\tau, z) = \theta_i(2\tau, 2z)$, then we obtain the following transformation laws.

$$(6)\quad \begin{cases} \varphi_2(-1/\tau, z/\tau) = \\ \qquad \sqrt{\dfrac{\tau}{2i}}\, e^{2\pi i z^2/\tau}(\varphi_3(\tau, z) - \varphi_2(\tau, z)), \text{ and} \\ \varphi_3(-1/\tau, z/\tau) = \\ \qquad \sqrt{\dfrac{\tau}{2i}}\, e^{2\pi i z^2/\tau}(\varphi_3(\tau, z) + \varphi_2(\tau, z)). \end{cases}$$

Our main theorem, which is straightforwardly proved by using the relation (6) is stated as follows.

Theorem 2.1. *Let* $f(x_1, y_1, x_2, y_2) \in R_n^G$ *where* $n \equiv 0\ (mod\ 8)$. *Then*

$$f(\varphi_3(\tau, 0), \varphi_2(\tau, 0), \varphi_3(\tau, z), \varphi_2(\tau, z))$$

is a Jacobi form of weight $n/2$ *and of index which is equal to the total degree of* f *in* x_2 *and* y_2.

The map defined in Theorem 2.1 is very useful in constructing many Jacobi forms. This map gives an injective ring homomorphism from R^G into $J_{4*,*}$. Here we give some typical examples. Many further examples will be discussed in subsequent papers.

Example 2.2. (i) $J(e_8, v_1 \mid X, Z) = 1 + X^4(7Z + 7) + X^8 Z$ with $w(v_1) = 1$, where e_8 is the extended [8,4,4] binary Hamming code and v_1 is any vector in V with $w(v_1) = 1$, corresponds to $E_{4,1}$, the unique Jacobi form (Eisenstein series) of weight 4 and index 1.

(ii) $\mathscr{E}_{12,1}(X, Z) = 1 - X^4(11Z + 22) - X^8(22Z + 11) + X^{12}Z$, which is a formal Jacobi polynomials in the sense of [9], i.e., the corresponding homogeneous polynomial $f(x_1, y_1, x_2, y_2)$ of degree 12 which is in R^H but not in R^G, corresponds to $E_{6,1}$, the unique Jacobi form (Eisenstein series) of weight 6 and index 1.

3. Concluding remarks. In this section we briefly mention how the map defined in Theorem 2.1 will be generalized further.

(3.1) If $f(x_1, y_1, x_2, y_2) \in R_n^H$ with $n \equiv 4(mod\ 8)$, then

$$f(\varphi_3(\tau, 0), \varphi_2(\tau, 0), \varphi_3(\tau, z), \varphi_2(\tau, z))$$

is a Jacobi form of weight $n/2$ and of index the total degree of f in x_2 and y_2.

Remark. It would be very interesting if one could construct Jacobi forms of odd weight by a similar method.

(3.2) A map similar to the map in Theorem 2.1 is defined also for ternary case. The resulting Jacobi forms thus obtained have indices k satisfying certain congruence conditions. The details will be discussed in a separate paper. Similar maps are also defined for other values of q.

(3.3) Multivariable Jacobi polynomials. Let $G = \langle \sigma_1, \sigma_2 \rangle \subset GL(2l, \boldsymbol{Z})$ with

$$\sigma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & & & & & \\ 1 & -1 & & & & & \\ & & 1 & 1 & & & \\ & & 1 & -1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & 1 \\ & & & & & 1 & -1 \end{pmatrix}, \text{ and}$$

$$\sigma_2 = \begin{pmatrix} \sqrt{-1} & & & & & & \\ & 1 & & & & & \\ & & \sqrt{-1} & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & \sqrt{-1} & \\ & & & & & & 1 \end{pmatrix}.$$

Let $R^G = \boldsymbol{C}[x_1, y_1, x_2, y_2, \ldots, x_l, y_l]^G$. Let $f \in R_n^G$ $(n \equiv 0\ (mod\ 8))$ with the total degree of f in x_i and y_i being $s_i (1 \le i \le l)$. Then for any nonnegative integers $m_1, m_2, \cdots, m_l$,

$$f(\varphi_3(\tau, m_1 z), \varphi_2(\tau, m_1 z), \varphi_3(\tau, m_2 z), \varphi_2(\tau, m_2 z), \cdots, \varphi_3(\tau, m_l z), \varphi_2(\tau, m_l z))$$

is a Jacobi form of weight $n/2$ and index $m = \sum_{i=1}^{l} m_i^2 s_i$. (Note that a similar result holds for $n \equiv 4\ (mod\ 8)$.)

Remark. It would be interesting to know the structure of $R^G = \boldsymbol{C}[x_1, y_1, x_2, y_2, \ldots, x_l, y_l]^G$ explicitly. In particular, it would be interesting to know how many of the Jacobi forms are constructed by applying (3.3). We hope and expect that the most of Jacobi forms (at least for weight $k \equiv 0\ (mod\ 4)$) are constructed this way.

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