Recurrence and Transience of Operator Semi-Stable Processes

By Gyeong Suck CHOI and Ken-iti SATO

Nagoya University

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1. Introduction and results. Operator semi-stable distributions on the d-dimensional Euclidean space $\boldsymbol{R}^d$ constitute a class of infinitely divisible distributions. They are studied by R. Jajte [4], [5], W. Krakowiak [6], A. Luczak [9], V. Chorny [2] and others. We call Lévy processes on $\boldsymbol{R}^d$ having operator semi-stable distributions at each time *operator semi-stable processes*. Here we mean by Lévy processes stochastically continuous processes with stationary independent increments starting at the origin. In this note we determine recurrence and transience of all non-degenerate operator semi-stable processes.

A distribution μ on $\boldsymbol{R}^d$ is called *operator semi-stable* if there exist a sequence $\{Y_n : n = 1,2,\cdots\}$ of i.i.d. (= independent identically distributed) random variables on $\boldsymbol{R}^d$, a strictly increasing sequence of positive integers k_n satisfying $k_{n+1}/k_n \to r$ with some $r \in [1, \infty)$, and sequences of invertible linear operators A_n acting in $\boldsymbol{R}^d$ and vectors b_n in $\boldsymbol{R}^d$ such that the distribution of

(1.1) $\quad A_n(Y_1 + Y_2 + \cdots + Y_{k_n}) + b_n$

weakly converges to μ as $n \to \infty$. R. Jajte [4] shows that if μ is full (that is, the support of μ is not contained in any $(d-1)$-dimensional hyperplane in $\boldsymbol{R}^d$), then a necessary and sufficient condition for μ to be operator semi-stable is that it is infinitely divisible and there exist a number $a \in (0, 1)$, a vector $b \in \boldsymbol{R}^d$, and an invertible linear operator A such that

(1.2) $$\mu^a = A\mu * \delta_b.$$

Here μ^a is the a-th convolution power of μ, $A\mu$ is the distribution defined by $A\mu(E) = \mu(A^{-1}E)$, and δ_b is the delta distribution at b. Using the relation (1.2), A. Luczak [9] and V. Chorny [2] describe the Lévy measure of μ.

In one dimension $(d = 1)$ A_n and A are simply multiplication by non-zero constants. P. Lévy [8], p. 204, introduced in one dimension the notion of semi-stability, which corresponded to the case $b = 0$ in (1.2), and determined their characteristic functions. R. Shimizu[13] made a study of relations of Lévy's semi-stability with limit theorems for sequences of i.i.d. random variables. V. M. Kruglov [7] studied the class of one-dimensional distributions which are limit distributions of $c_n(Y_1 + Y_2 + \cdots + Y_{k_n}) + b_n$ for i.i.d. random variables $\{Y_n\}$ with $c_n > 0$, b_n real, and $k_{n+1}/k_n \to r \in [1, \infty)$. In general finite dimensions, if $k_n = n$ and A_n is a positive constant multiple of the identity operator for each n, then the definition above of operator semi-stability gives the class of stable distributions on $\boldsymbol{R}^d$. If $k_n = n$, then the definition above gives the class of operator stable distributions, which were first introduced by M. Sharpe [12]. On the other hand, if A_n is a non-zero constant multiple of the identity operator for each n, then the limit distributions are called semi-stable. The class of operator semi-stable distributions extends these classes. The corresponding Lévy processes are called stable processes, operator stable processes, semi-stable processes, and operator semi-stable processes, respectively. Classification of stable processes into recurrent and transient is well-known. It is extended in [1] to semi-stable processes. Operator stable processes are discussed in [10], but their recurrence and transience are not treated.

Our result is as follows. We say that a Lévy process is *non-degenerate* if its distribution at each $t > 0$ is full.

Theorem. *Let $\{X_t\}$ be a non-degenerate operator semi-stable process on the plane $\boldsymbol{R}^2$. If $\{X_t\}$ is not Gaussian, then it is transient.*

Note that, for $d > 3$, all non-degenerate Lévy processes on $\boldsymbol{R}^d$ are transient (see [11] for proof). Also note that operator semi-stable processes on the line $\boldsymbol{R}^1$ are semi-stable processes in the sense of [1], and their classification is obtained in [1]. A Gaussian Lévy process on the plane $\boldsymbol{R}^2$ is recurrent or transient according as its mean is zero or non-zero, respectively. Therefore our theorem completes classification of operator semi-stable processes into recurrent and transient.

2. Proof of Theorem. Let μ be a full operator semi-stable distribution on $\boldsymbol{R}^d$. Then, as stated in the preceding section, there exist a, b, and A satisfying the relation (1.2). It is shown by R. Jajte [4] that μ is in one of the three cases below (Cases 1-3).

Case 1: Every eigenvalue λ of A satisfies $|\lambda| < \sqrt{a}$ and μ has no Gaussian part.

Case 2: Every eigenvalue λ of A satisfies $|\lambda| = \sqrt{a}$ and is a simple root of the minimal polynomial of A, and μ is Gaussian.

Case 3: There are two A-invariant proper subspaces V_1 and V_2 satisfying $\boldsymbol{R}^d = V_1 \oplus V_2$ and μ is decomposed into $\mu = \mu_1 * \mu_2$ such that μ_1 and μ_2 are concentrated on V_1 and V_2, respectively, $\mu_1|_{V_1}$ is a full operator semi-stable distribution on V_1 without Gaussian component, $\mu_2|_{V_2}$ is a full Gaussian distribution on V_2, the eigenvalues of $A|_{V_1}$ have absolute values $< \sqrt{a}$ and the eigenvalues of $A|_{V_2}$ are simple roots of the minimal polynomial of A and have absolute values $= \sqrt{a}$.

Denote the Euclidean inner product of x, $y \in \boldsymbol{R}^d$ by $\langle x, y \rangle$, the Euclidean norm of x by $|x|$, and the operator norm of a linear operator B acting in $\boldsymbol{R}^d$ by $\|B\|$. The adjoint operator of B is denoted by B'. Given an invertible linear operator B with $\|B\| < 1$, let

(2.1) $$G = \{x \in \boldsymbol{R}^d : |x| \leq 1 \text{ and } |B^{-1}x| > 1\}.$$

This is the set employed by A. Łuczak [9]. It is not hard to show that $B^n G$ and $B^m G$ are disjoint if n and m are distinct integers, and that

(2.2) $$\{x \in \boldsymbol{R}^d : 0 < |x| \leq 1\} = \bigcup_{n=0}^{\infty} B^n G,$$
$$\{x \in \boldsymbol{R}^d : |x| > 1\} = \bigcup_{n=1}^{\infty} B^{-n} G.$$

Let $\hat{\mu}(z)$, $z \in \boldsymbol{R}^d$, be the characteristic function of μ. Let $\psi(z)$ be the continuous function on $\boldsymbol{R}^d$ such that $\hat{\mu}(z) = e^{\psi(z)}$ and $\psi(0) = 0$. Let $K(z) = \mathrm{Re}(-\psi(z))$. By the Lévy process analogue of the Chung-Fuchs criterion [3] of recurrence and transience for random walks, the Lévy process $\{X_t\}$ with the distribution μ at $t = 1$ is transient if and only if

(2.3) $$\limsup_{\alpha \downarrow 0} \int_{|z|<\varepsilon} \mathrm{Re}\left(\frac{1}{\alpha - \psi(z)}\right) dz < \infty$$

for some $\varepsilon > 0$ (see [11] for proof). If

(2.4) $$\int_{|z|<\varepsilon} \frac{dz}{K(z)} < \infty$$

for some $\varepsilon > 0$, then (2.3) follows and $\{X_t\}$ is transient, because

$$\mathrm{Re}\left(\frac{1}{\alpha - \psi(z)}\right) \leq \frac{1}{|\alpha - \psi(z)|} \leq \frac{1}{\alpha + K(z)} \leq \frac{1}{K(z)}.$$

Now let $d = 2$ and proceed to the proof of our theorem. Case 2 is excluded from the theorem.

Suppose that μ is in Case 1. The relation (1.2) is expressed as

$$\hat{\mu}(z)^a = \hat{\mu}(A'z) e^{i\langle b, z \rangle}.$$

Hence we have

(2.5) $$aK(z) = K(A'z).$$

We claim that there are $p < 2$ and $c > 0$ such that

(2.6) $$K(z) \geq c|z|^p$$

in a neighborhood of 0. By Remark 1.1 of A. Łuczak [9] we may and do assume that $\|A\| < 1$. Since we are in Case 1, we may and do assume that the eigenvalues λ_1, λ_2 of A satisfy $0 < |\lambda_1| \leq |\lambda_2| < \sqrt{a}$. Choose $p < 2$ such that $|\lambda_2| < a^{1/p}$. Let $B = A'$ and define the set G by (2.1). By Lemma 2.1 of A. Łuczak [9], $K(z) > 0$ for $z \neq 0$. Since the closure of G is compact and does not contain 0, we can find $c > 0$ such that $K(z) \geq c|z|^p$ for $z \in G$. Since $|\lambda_2|$ is the spectral radius of B, we have $\|B^n\|^{1/n} \to |\lambda_2|$ as $n \to \infty$. Hence there is n_0 such that $\|B^n\|^{1/n} < a^{1/p}$ for $n \geq n_0$. Now, it follows from (2.5) that, for any $z \in G$ and $n \geq n_0$,

$$K(B^n z) = a^n K(z)$$
$$\geq ca^n |z|^p > c\|B^n\|^p |z|^p \geq c|B^n z|^p.$$

Thus

$$K(z) \geq c|z|^p \text{ for } z \in \bigcup_{n=n_0}^{\infty} B^n G.$$

Noting (2.2), we can choose $\varepsilon > 0$ such that $\{z : 0 < |z| < \varepsilon\}$ is contained in $\cup_{n=n_0}^{\infty} B^n G$. Hence (2.6) is proved for z satisfying $|z| < \varepsilon$. Since $d = 2$, the estimate (2.6) with $p < 2$ implies (2.4), which proves transience.

Suppose that μ is in Case 3. The subspaces V_1, V_2 in the statement of Case 3 are one-dimensional. Hence, for $j = 1$ and 2, $A|_{V_j} = \lambda_j I_{V_j}$ with some $\lambda_j \in \boldsymbol{R}$, where I_{V_j} is the identity operator on V_j. These λ_1 and λ_2 are eigenvalues of A, and hence $0 < |\lambda_1| < |\lambda_2| = \sqrt{a}$. Let $V_j^{\perp}$ be the orthogonal complement of V_j. Then $\boldsymbol{R}^2 = V_2^{\perp} \oplus$

$V_1^{\perp}$. Fix z_1^0 and z_2^0 with norm 1 in $V_2^{\perp}$ and $V_1^{\perp}$, respectively. Any $z \in \boldsymbol{R}^2$ is represented as $z = \zeta_1 z_1^0 + \zeta_2 z_2^0$ with $\zeta_1, \zeta_2 \in \boldsymbol{R}$. Using μ_1 and μ_2 in the statement of Case 3, we have

$$\hat{\mu}(z) = \widehat{\mu_1}(\zeta_1 z_1^0)\widehat{\mu_2}(\zeta_2 z_2^0),$$
$$(\mu_j|_{V_j})^a = (\lambda_j I_{V_j})(\mu_j|_{V_j}) * \delta_{b_j} \text{ for } j = 1,2$$

with some $b_j \in V_j$. Choose $p < 2$ such that $|\lambda_1| < a^{1/p}$. Then, by the same argument as in Case 1, we see that there exists $c_1 > 0$ such that $-\log|\widehat{\mu_1}(\zeta_1 z_1^0)| \geq c_1|\zeta_1|^p$ for ζ_1 sufficiently close to 0. (More precisely, as is shown in [1] and [8], there exist $0 < c_1' \leq c_2'$ such that $c_1'|\zeta_1|^{\alpha} \leq -\log|\widehat{\mu_1}(\zeta_1 z_1^0)| \leq c_1''|\zeta_1|^{\alpha}$ for all ζ_1, where α is defined by $|\lambda_1| = a^{1/\alpha}$.) Since $\mu_2|_{V_2}$ is Gaussian, $-\log|\widehat{\mu_2}(\zeta_2 z_2^0)| = c_2\zeta_2^2$ with some $c_2 > 0$. Since $K(z) = -\log|\hat{\mu}(z)|$, we see that there is $\varepsilon > 0$ such that

$$K(\zeta_1 z_1^0 + \zeta_2 z_2^0) \geq c_1|\zeta_1|^p + c_2\zeta_2^2 \tag{2.7}$$

for $|\zeta_1| < \varepsilon$ and $|\zeta_2| < \varepsilon$. Choose $\varepsilon' > 0$ such that $|z| < \varepsilon'$ implies $|\zeta_1| < \varepsilon$ and $|\zeta_2| < \varepsilon$. As the Lebesgue measure dz equals $cd\zeta_1 d\zeta_2$ with some positive constant c, we get, using (2.7),

$$\begin{aligned}
&\int_{|z|<\varepsilon'} \frac{dz}{K(z)} \\
&\leq c\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{d\zeta_1 d\zeta_2}{K(\zeta_1 z_1^0 + \zeta_2 z_2^0)} \\
&\leq c\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{d\zeta_1 d\zeta_2}{c_1|\zeta_1|^p + c_2\zeta_2^2} \\
&\leq \frac{2c}{\sqrt{c_1c_2}}\int_{-\varepsilon}^{\varepsilon} |\zeta_1|^{-p/2} \arctan\left(\sqrt{\frac{c_2}{c_1}}\,|\zeta_1|^{-p/2}\varepsilon\right) d\zeta_1 \\
&\leq \frac{\pi c}{\sqrt{c_1c_2}}\int_{-\varepsilon}^{\varepsilon} |\zeta_1|^{-p/2}\, d\zeta_1 \\
&< \infty.
\end{aligned}$$

Hence (2.4) is obtained. This implies transience. Proof is complete.

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