A Note on the Extremality of Teichmüller Mappings

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Introduction. For a hyperbolic Riemann surface R, we denote by $A_2(R)$ the set of all holomorphic quadratic differentials $\phi = \phi(z)dz^2$ on R, and set

$$A_2^p(R) := \left\{\phi \in A_2(R) : \| \phi \|_p := \left(\int_R \lambda_R^{2-2p} | \phi |^p\right)^{1/p} < \infty\right\} \text{ for } 1 \leq p < \infty,$$

$$A_2^\infty(R) := \{\phi \in A_2(R) : \| \phi \|_\infty := \text{ess sup}_R \lambda_R^{-2} | \phi | < \infty\},$$

where $\lambda_R = \lambda_R(z) | dz |$ is the hyperbolic metric on R with constant negative curvature -4. For simplicity, we often write $\| \phi \|_{p,E}$ instead of $\left(\int_E \lambda_R^{2-2p} | \phi |^p\right)^{1/p}$.

A quasiconformal mapping f of a Riemann surface R is called *extremal* if it has the smallest maximal dilatation in the class Q_f of all quasiconformal mappings of R which are homotopic to f relative to the border ∂R of R. An extremal mapping is called *uniquely extremal* if there are no other extremal mappings in Q_f. Hamilton, Reich and Strebel have characterized the extremality: a quasiconformal mapping f is extremal if and only if there is a sequence $\{\phi_n\}_{n=1}^\infty$ in $A_2^1(R)$, $\| \phi_n \|_1 = 1$, such that $\lim_{n\to\infty} \int_R \mu_f \phi_n = \text{ess sup}_R | \mu_f |$, where μ_f is the Beltrami coefficient of f (Strebel [10]). Such a sequence is called a *Hamilton sequence* for f, and it is said to *degenerate* if it weakly converges to 0.

A quasiconformal mapping whose Beltrami coefficient has the form $k\bar{\phi}/| \phi |$, where $0 \leq k < 1$ and $\phi \in A_2(R)\setminus \{0\}$, is called a *Teichmüller mapping* corresponding to ϕ. In the theory of extremal quasiconformal mappings, Teichmüller mappings play an important role. We know that every Teichmüller mapping corresponding to $\phi \in A_2^1(R)$ is uniquely extremal (Strebel [10]), but there are non-extremal, and extremal but not uniquely extremal Teichmüller mappings (Strebel [8]). So it is expected to find conditions for a holomorphic quadratic differential ϕ that guarantees the Teichmüller mapping corresponding to ϕ to be extremal or not. For the case R is the unit disk $\boldsymbol{D}$, some extremality theorems have been proved, for instance, Sethares [7], Reich-Strebel [6], Hayman-Reich [2] and one of the authors [3]. On the other hand, Strebel [9] has constructed an example which shows that a lift to the universal covering of an extremal Teichmüller mapping of a compact Riemann surface is not necessarily extremal, and recently McMullen [4] and one of the authors [5] have generalized this.

1. In the present paper, we prove the following:

Theorem 1. *Suppose that R is a hyperbolic Riemann surface of finite analytic type, and that $\pi : \tilde{R} \to R$ is an infinite sheeted regular (i.e. unbounded and unramified) covering from another Riemann surface $\tilde{R}$ to R which satisfies the condition:*

(*) *for any puncture a of R and any cusped neighborhood V of a, there is an integer m such that the restriction of π to any connected component of $\pi^{-1}(V)$ is at most m sheeted.*

Then for $\Psi \in A_2^\infty(R)$, $\Psi \neq 0$, and $\phi \in \cup_{1\leq p<\infty} A_2^p(\tilde{R})$, the Teichmüller mapping $f_{\pi^*\Psi}$ corresponding to the pull-back $\pi^*\Psi \in A_2^\infty(\tilde{R})$ and the Teichmüller mapping $f_{\pi^*\Psi+\phi}$ corresponding to $\pi^*\Psi + \phi \in A_2^\infty(\tilde{R})$ have the same Hamilton sequences. In particular, $f_{\pi^*\Psi}$ is extremal if and only if so is $f_{\pi^*\Psi+\phi}$.

As an application of our Theorem 1 and McMullen's theorem, we have

Corollary 1. *Let $\pi : \tilde{R} \to R$ be a covering as in Theorem 1. If, moreover, π is nonamenable, then for any $\Psi \in A_2^\infty(R)\setminus \{0\}$ and any $\phi \in A_2^p(\tilde{R})$, $1 \leq p < \infty$, any lifts to the unit disk of the Teichmüller mapping of $\tilde{R}$ corresponding to $\pi^*\Psi + \phi$ are not extremal.*

Proof. By McMullen's theorem [4], the

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Teichmüller mapping corresponding to $\pi^*\Psi$ is not extremal. Thus the Teichmüller mapping corresponding to $\pi^*\Psi+\phi$ is not extremal by Theorem 1, hence its lifts to the unit disk are not extremal.

For a Fuchsian group Γ acting on the unit disk $\boldsymbol{D}$, define

$A_2^\infty(\boldsymbol{D},\Gamma):=\{\phi\in A_2^\infty(\boldsymbol{D}):\gamma^*\phi=\phi \text{ for all } \gamma\in\Gamma\}$.

Corollary 2. *If Γ is a torsion-free Fuchsian group acting on $\boldsymbol{D}$ such that the Riemann surface $\Gamma\backslash\boldsymbol{D}$ is compact, then for any $\Psi\in A_2^\infty(\boldsymbol{D},\Gamma)\setminus\{0\}$ and any $\phi\in A_2^p(\boldsymbol{D})$, $1\le p<\infty$, the Teichmüller self-mapping of $\boldsymbol{D}$ corresponding to $\Psi+\phi$ is not extremal.*

In particular, there is a non-extremal Teichmüller mapping which is not compatible with any nontrivial Fuchsian groups.

To prove Theorem 1, we need some lemmas. The hyperbolic distance between $a, b\in R$ is denoted by $d_R(a,b)$. For $a\in R$ and $l>0$, we set $\Delta(a;l):=\{b\in R: d_R(b,a)<l\}$. The supremum of all $l>0$ for which $\Delta(a;l)$ is simply connected is called *injectivity radius* at a, and denoted by $\operatorname{inj\,rad}(a)$.

First of all, by the mean-value theorem for holomorphic functions and Hölder's inequality, we have

Lemma 1. *Suppose that R is a hyperbolic Riemann surface and the injectivity radius at $a\in R$ is not less than l. Then for all $\phi\in A_2(R)$ and $1\le p<\infty$,*

$$(\lambda_R^{-2}|\phi|)(a)\le\frac{1}{(\pi\tanh^2 l)^{1/p}}\|\phi\|_{p,\Delta(a;l)}.$$

Lemma 2. *Let $\pi:\tilde R\to R$ be a regular covering of a hyperbolic Riemann surface R, and l_0 be the injectivity radius at $a\in R$. Then for $\phi\in A_2^1(\tilde R)$ and $0<l\le l_0/2$, we have*

$\|\phi\|_{1,\pi^{-1}(\Delta(a;l))}\le\|\phi\|_1\tanh^2 l/\tanh^2(l_0/2)$.

Proof. Let $\tilde a\in\pi^{-1}(a)$ and $b\in\Delta(\tilde a;l)$. Since the injectivity radius at b is not less than $l_0/2$, we see $(\lambda_{\tilde R}^{-2}|\phi|)(b)\le\|\phi\|_{1,\Delta(\tilde a;l_0)}/(\pi\tanh^2(l_0/2))$ by Lemma 1. Integrating this on $\Delta(\tilde a;l)$ and summing with respect to $\tilde a$, we obtain Lemma 2.

Lemma 3. *Let $\pi:\tilde R\to R$ be a regular covering of a hyperbolic Riemann surface R, a be a puncture of R, V be a cusped neighborhood of a which is expressed by $\{0<|z|<1\}$ in terms of a local parameter z, and $\cup_j\tilde V_j$ be the decomposition of $\pi^{-1}(V)$ to its connected components. If there is an integer m such that the numbers of sheets of the restrictions $\pi|_{\tilde V_j}$ are bounded by m, then we have $\|\phi\|_{\pi^{-1}(\{0<|z|<r\})}\le C(m)\|\phi\|_1 r^{1/m}$ for $\phi\in A_2^1(\tilde R)$ and $0<r\le 1/3$, where $C(m)$ is a constant depending only on m.*

Proof. Take a local parameter ζ on $\tilde V_j$ in terms of which $\pi(\zeta)=\zeta^n$, where n is the number of sheets of the covering $\pi|_{\tilde V_j}:\tilde V_j\to V$. Since $\phi=\phi(\zeta)d\zeta^2$ has at most a simple pole at $\zeta=0$, by applying the mean-value theorem to $\zeta\phi(\zeta)$, we have $|\phi(\zeta)|\le C_0(m)\|\phi\|_{1,\tilde V_j}/|\zeta|$ for $0<|\zeta|<(1/3)^{1/n}$, from which the assertion follows by the same way as in Lemma 2.

Proof of Theorem 1. Because $\pi^*\Psi$, $\pi^*\Psi+\phi\notin A_2^1(\tilde R)$, all Hamilton sequences for $f_{\pi^*\Psi}$ and for $f_{\pi^*\phi+\phi}$, if any, must degenerate. So it is enough to show that

$$(1)\quad\lim_{n\to\infty}\int_{\tilde R}|\phi_n|\left|\frac{\pi^*\Psi}{|\pi^*\Psi|}-\frac{\pi^*\Psi+\phi}{|\pi^*\Psi+\phi|}\right|=0$$

for any sequence $\{\phi_n\}_{n=1}^\infty\subset A_2^1(\tilde R)$, $\|\phi_n\|=1$, which is weakly convergent to 0.

Let $\varepsilon>0$ be a small number. Let $a_1,\dots,a_k$ be the punctures of R, and $b_1,\dots,b_l\in R$ be the zeros of Ψ, and take small cusped neighborhoods $V_1,\dots,V_k$ of $a_1,\dots,a_k$, and small disks $U_1,\dots,U_l$ centered on $b_1,\dots,b_l$ so that they are mutually disjoint. Set $N:=\cup_{j=1}^k V_j\cup\cup_{j=1}^l U_j$, and let δ be the minimum value of $\lambda_R^{-2}|\Psi|$ on $R\setminus N$. By Lemmas 2 and 3, we may assume that $\|\phi_n\|_{1,\pi^{-1}(N)}<\varepsilon$ for any n. Take a large compact set $K\subset\tilde R$ so that $\lambda_{\tilde R}^{-2}|\phi|\le\varepsilon\delta$ outside $K\cup\pi^{-1}(N)$. By Lemma 1, we can take such a K. Since $|\phi|/|\pi^*\Psi|\le\varepsilon$ on $\tilde R\setminus(K\cup\pi^{-1}(N))$, we have

$$\int_{\tilde R}|\phi_n|\left|\frac{\pi^*\Psi}{|\pi^*\Psi|}-\frac{\pi^*\Psi+\phi}{|\pi^*\Psi+\phi|}\right|\le$$

$$\int_{\tilde R\setminus(K\cup\pi^{-1}(N))}2\varepsilon|\phi_n|+\int_K 2|\phi_n|+\int_{\pi^{-1}(N)}2|\phi_n|$$

$$\le 2\varepsilon+2\int_K|\phi_n|+2\varepsilon.$$

Letting $n\to\infty$ and $\varepsilon\to 0$, we obtain (1), and the theorem is proved.

2. To prove (1), the condition (∗) is essential. In fact, we can show

Theorem 2. *Let R be a (not necessarily analytically finite) Riemann surface with a puncture a, V be a cusped neighborhood of a, $\pi:\tilde R\to R$ be a regular covering, and $\{\tilde V_j\}_j$ be the connected components of $\pi^{-1}(V)$. If the numbers of sheets of the coverings $\pi|_{\tilde V_j}:\tilde V_j\to V$ are unbounded, then there* *exist* $\phi \in A_2^1(\tilde{R})$ *and a sequence* $\{\phi_n\}_{n=1}^{\infty} \subset A_2^1(\tilde{R})$, $\|\phi_n\| = 1$, *such that for an arbitrary* $\Psi \in A_2^{\infty}(R)$, $0 < \|\Psi\|_{\infty} \leq 1$,

$$\lim_{n\to\infty} \int_{\tilde{R}} \frac{\overline{\pi^*\Psi}}{|\pi^*\Psi|} \phi_n = 0, \quad but$$

$$\lim_{n\to\infty} \int_{\tilde{R}} \frac{\overline{\pi^*\Psi + \bar{\phi}}}{|\pi^*\Psi + \phi|} \phi_n = 1.$$

Lemma 4. *Let* R *be a Riemann surface, and* $a \in R$. *If* $\operatorname{inj\,rad}(a) \geq 2l$, $l \geq l_0 := \log(\sqrt{2} + 1)$, *then there is* $\phi \in A_2^1(R)$ *such that* $\|\phi\|_1 = 1$,

$$\|\phi\|_{1,R\setminus\Delta(a;l)} \leq 2^{-1}(1 - \tanh^2 l),$$

$$\lambda_R^{-2} |\phi| \geq 2^{-5}(1 - \tanh^2 l)^2 \quad on\ \Delta(a; l).$$

Moreover, let b *be a point on* R *for which* $\operatorname{inj\,rad}(b) \geq l_0$ *and* $d_R(b, a) \geq l' + l_0$, *then*

$$(\lambda_R^{-2} |\phi|)(b) \leq 1 - \tanh^2 l'.$$

Proof. Let $\pi : \boldsymbol{D} \to R$ be a universal covering such that $\pi(0) = a$, Γ be its covering transformation group. Then, by the standard argument and Lemma 1, it is not difficult to see that $\phi := (\pi^*)^{-1}(\sum_{\gamma\in\Gamma}(\gamma')^2 / \|\sum_{\gamma\in\Gamma}(\gamma')^2\|_1)$ has the properties in Lemma 4.

Proof of Theorem 2. We may assume that $V = \{0 < |z| < e^{-2\pi}\}$ and $\lambda_R(z)\,|dz| = (2|z|\,|\log|z||)^{-1}|dz|$ in terms of a local parameter z. Since each $\Psi = \Psi(z)dz^2$ has at most a simple pole at a, we have $\lambda_R^{-2}(z)\,|\Psi(z)| \leq C_1 |z|\,|\log|z||^2$, where C_1 is a universal constant.

Let $\{l_n\}_{n=1}^{\infty}$ be a sequence such that $l_n \geq l_0$ and $\lim l_n = \infty$, and define a sequence of large numbers $\{l_n'\}_{n=1}^{\infty}$ so that $1 - \tanh^2 l_n' \leq 2^{-(2n+7)}(1 - \tanh^2 l_n)^2$. Our assumption on the numbers of sheets of the coverings implies that we can take disks $\Delta_n' := \Delta(a_n; l_n + l_n')$ in $\pi^{-1}(\{C_1 |z|\,|\log|z||^2 \leq 2^{-(2n+7)}(1 - \tanh^2 l_n)^2\})$. We may assume that these disks $\{\Delta_n'\}_{n=1}^{\infty}$ are mutually disjoint. Let $\phi_n \in A_2^1(\tilde{R})$ be the holomorphic quadratic differentials obtained by applying Lemma 4, and set $\phi := \sum_{n=1}^{\infty} 2^{-n}\phi_n \in A_2^1(\tilde{R})$. Since $\lambda_{\tilde{R}}^{-2}|\pi^*\Psi| \leq 2^{-(2n+2)}\lambda_{\tilde{R}}^{-2}|\phi_n|$ and $\lambda_{\tilde{R}}^{-2}|\sum_{k\neq n} 2^{-k}\phi_k| \leq 2^{-(2n+2)}\lambda_R^{-2}|\phi_n|$ on $\Delta_n := \Delta(a_n; l_n)$, we have

$$|(\pi^*\Psi + \phi)/|\pi^*\Psi + \phi| - \phi_n/|\phi_n|| \leq 2^{-n}.$$

Thus we see

$$\left|1 - \int_{\tilde{R}} \frac{\overline{\pi^*\Psi + \bar{\phi}}}{|\pi^*\Psi + \phi|} \phi_n\right|$$

$$\leq \left|1 - \int_{\Delta_n} \frac{\overline{\phi_n}}{|\phi_n|} \phi_n\right| + \left|\int_{\Delta_n} \left(\frac{\overline{\pi^*\Psi + \bar{\phi}}}{|\pi^*\Psi + \phi|} - \frac{\overline{\phi_n}}{|\phi_n|}\right) \phi_n\right| + \left|\int_{\tilde{R}\setminus\Delta_n} \frac{\overline{\pi^*\Psi + \bar{\phi}}}{|\pi^*\Psi + \phi|} \phi_n\right|$$

$$\leq 1 - \int_{\Delta_n} |\phi_n| + \frac{1}{2^n}\int_{\Delta_n} |\phi_n| + \int_{\tilde{R}\setminus\Delta_n} |\phi_n| \to 0$$

as $n \to \infty$.

On the other hand, there is a constant C_2 such that $\|\Phi\|_{1,V} \leq C_2 \|\Phi\|_{1,R\setminus V}$ for any $\Phi \in A_2^1(R)$. Hence

$$\left|\int_{\tilde{R}} \frac{\overline{\pi^*\Psi}}{|\pi^*\Psi|} \phi_n\right| = \left|\int_R \frac{\bar{\Psi}}{|\Psi|} \Theta_{R\setminus\tilde{R}}\phi_n\right| \leq (C_2 + 1)$$

$$\times \int_{R\setminus V} |\Theta_{R\setminus\tilde{R}}\phi_n| \leq (C_2 + 1) \int_{\tilde{R}\setminus\Delta_n} |\phi_n| \to 0,$$

where $\Theta_{R\setminus\tilde{R}} : A_2^1(\tilde{R}) \to A_2^1(R)$ is the relative Poincaré series operator. This completes the proof.

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