On the Difinition of the Virtanen Property for Riemannian Manifolds

By Mitsuru NAKAI

Department of Mathematics, Nagoya Institute of Technology

(Communicated by Kiyosi ITÔ, M. J. A., Jan. 12, 1995)

In our former paper [4] we introduced a notion which we called the *Virtanen property* for Riemannian manifolds. The property is always fulfilled by two dimensional Riemannian manifolds so that it often ensures the possibility of extending certain potential theoretic results valid for two dimensional case to higher dimensions. The purpose of this paper is to give a new definition of the Virtanen property which is equivalent to but more understandable than that given in [4].

Throughout this paper we let M be a noncompact, connected and orientable Riemannian manifold of class C^∞ of dimension $n \geqq 2$. Let (g_{ij}) be the metric tensor on M and $(g^{ij}) = (g_{ij})^{-1}$. With an s-form α on M $(0 \leqq s \leqq n)$ whose local expression in a local parameter $x = (x^1, \ldots, x^n)$ is

$$\alpha = \sum_{i_1 < \cdots < i_s} a_{i_1 \cdots i_s}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_s}$$

we associate a nonnegative function $|\alpha|$ on M, usually referred to as the *point norm* of α, given by

$$|\alpha|^2 = \sum_{i_1 < \cdots < i_s} \Big(\sum_{j_1, \cdots, j_s} g^{i_1 j_1} \cdots g^{i_s j_s} a_{j_1 \cdots j_s} \Big) a_{i_1 \cdots i_s}. \tag{1}$$

If α is measurable, then we can consider its p-norm $(1 \leqq p \leqq \infty)$

$$\| \alpha \|_p = \Big(\int_M |\alpha|^p dV \Big)^{1/p} \ (1 \leqq p < \infty),$$

$$\| \alpha \|_\infty = \operatorname{ess.\,sup}_M |\alpha|,$$

where dV is the volume element on M. Using these notations we can give our new definition of the Virtanen property:

Definition. *The manifold M is said to possess the Virtanen property if for any C^∞ $(n-2)$-form α on M with $\| d\alpha \|_2 < \infty$ there exists a sequence (α_m) of C^∞ $(n-2)$-forms α_m on M such that*

$$\| \alpha_m \|_\infty < \infty \ (m = 1, 2, \ldots), \tag{2}$$

$$\lim_{m \to \infty} \| d\alpha - d\alpha_m \|_2 = 0. \tag{3}$$

If the dimension $n = 2$, then the given form α and sought forms α_m are 0-forms, i.e. functions, on M. Taking α_m as a suitable regularization of the function $\max(\min(\alpha, m), -m)$ for each $m = 1, 2, \ldots$, we see that (2) and (3) are satisfied by these α and α_m (cf. e.g. [3]) so that the Virtanen property is always possessed by any two dimensional Riemannian manifold M. In our former definition of the Virtanen property in [4] we had

$$N[\alpha_m] := \sup\{ \| \alpha_m \wedge d\varphi \|_2 / \| d\varphi \|_2 : \varphi \in C_0^\infty(M) \setminus \{0\} \} < \infty \tag{4}$$

instead of (2). The function norm $\| \alpha \|_\infty$ is much easier to compute than the operator norm $N[\alpha]$ so that we may say that our new definition is better than our former one. To assure that these two definitions are actually equivalent we have to prove that these two norms are equivalent. The practical purpose of this paper is, thus, to prove the following

Theorem. *The norms $\| \alpha \|_\infty$ and $N[\alpha]$ for any C^∞ $(n-2)$-form α are equivalent, i.e. the following inequalities are valid for every C^∞ $(n-2)$-form α on M :*

$$(n/2)^{-1/2} \| \alpha \|_\infty \leqq N[\alpha] \leqq \| \alpha \|_\infty. \tag{5}$$

Observe that (5) implies $N[\alpha] = \| \alpha \|_\infty$ for $n = 2$, which we already remarked in [4]. Inequalities in (5) are *sharp* in the following sense: for every dimension $n \geqq 2$, there is a couple (M, α) such that $N[\alpha] = (n/2)^{-1/2} \| \alpha \|_\infty > 0$ and also $N[\alpha] = \| \alpha \|_\infty > 0$. Such examples will be given right after the proof of (5).

Proof of Theorem. A parametric neighborhood $(U ; x)$ at $\xi \in M$ is always supposed to satisfy $x(\xi) = 0$. We say that a parametric neighborhood $(U ; x)$ at $\xi \in M$ is *special* if the components of the metric tensor $(g_{ij}(x))$ in the local parameter x takes the form $g_{ij}(0) = \delta_{ij}$ so that $g^{ij}(0) = \delta^{ij}$ as well $(i, j = 1, \ldots, n)$.

We start with the proof for the second inequality in (5) which is simple. Take any $\varphi \in$

1991 *Mathematics Subject Classification.* 31C12.

This work was partly supported by Grant-in-Aid for Scientific Research, No. 06640227, Japanese Ministry of Education, Science and Culture.

$C_0^\infty(M)\setminus\{0\}$ and any point $\xi \in M$. We can find a special parametric neighborhood $(U\,;x)$ at $\xi \in M$ (cf. e.g. [1]). Let the local expression of α in terms of x be

(6) $\alpha = \sum_{i_1<\cdots<i_{n-2}} a_{i_1\cdots i_{n-2}}(x)dx^{i_1}\wedge\cdots\wedge dx^{i_{n-2}}$.

On denoting $D_j\varphi = \partial\varphi/\partial x^j$ we have

(7) $\alpha\wedge d\varphi = \sum_{j=1}^n(\sum^j_{i_1<\cdots<i_{n-2}} a_{i_1\cdots i_{n-2}}(x)D_j\varphi(x)dx^{i_1}\wedge\cdots\wedge dx^{i_{n-2}}\wedge dx^j)$

where $\sum^j_{i_1<\cdots<i_{n-2}}$ means the sum with respect to $i_1\cdots i_{n-2}$ such that $\{i_1,\ldots,i_{n-2}\}\subset\{1,\ldots,n\}\setminus\{j\}$ and $i_1<\cdots<i_{n-2}$. In view of (1) and $g^{ij}(0)=\delta^{ij}$ we deduce that

$$|\alpha|(\xi)^2 = \sum_{i_1<\cdots<i_{n-2}} a_{i_1\cdots i_{n-2}}(0)^2,\ |d\varphi|(\xi)^2 = \sum_{j=1}^n (D_j\varphi(0))^2,$$

$$|\alpha\wedge d\varphi|(\xi)^2 = \sum_{j=1}^n\Big(\sum^j_{i_1<\cdots<i_{n-2}} a_{i_1\cdots i_{n-2}}(0)^2(D_j\varphi(0))^2\Big).$$

Since $\sum^j_{i_1<\cdots<i_{n-2}} a_{i_1\cdots i_{n-2}}(0)^2 \leqq |\alpha|(\xi)^2 \leqq \|\alpha\|_\infty^2$, we see that

$$|\alpha\wedge d\varphi|(\xi)^2 = \sum_{j=1}^n\Big\{\sum^j_{i_1<\cdots<i_{n-2}} a_{i_1\cdots i_{n-2}}(0)^2\Big\}(D_j\varphi(0))^2$$
$$\leqq \|\alpha\|_\infty^2\sum_{j=1}^n(D_j\varphi(0))^2 = \|\alpha\|_\infty^2|d\varphi|(\xi)^2,$$

i.e. we have obtained that $|\alpha\wedge d\varphi|^2 \leqq \|\alpha\|_\infty^2|d\varphi|^2$ on M. Hence we have $\|\alpha\wedge d\varphi\|_2 \leqq \|\alpha\|_\infty\|d\varphi\|_2$, which proves $N[\alpha]\leqq\|\alpha\|_\infty$.

We turn to the proof for the first inequality in (5) which is less simple. Fix any point $\xi\in M$ and choose a special parametric neighborhood $(U\,;x)$ at ξ. Again let (6) be the local expression of α in the local parameter x in U. Let B_δ be the open ball of radius $\delta>0$ centered at the origin of the Euclidean n-space and $U_\delta=\{\eta\in U: |x(\eta)|<\delta\}$. There is a $\delta_0>0$ such that $\bar U_\delta$ is compact in M and $x:\bar U_\delta\to\bar B_\delta$ is a homeomorphism for $0<\delta<\delta_0$. Fix two arbitrary real numbers $K>1$ and $\varepsilon>0$. We can find and then fix a $0<\delta<\delta_0$ such that for every $x\in B_\delta$

(8) $K^{-1}(\delta_{ij})\leqq(g_{ij}(x))\leqq K(\delta_{ij})$ and $K^{-1}(\delta^{ij})\leqq(g^{ij}(x))\leqq K(\delta^{ij})$

in the sense of matrix inequalities and also for every $x\in B_\delta$

(9) $a_{i_1\cdots i_{n-2}}(0)^2-\varepsilon < a_{i_1\cdots i_{n-2}}(x)^2$

for every $\{i_1,\ldots,i_{n-2}\}\subset\{1,\ldots,n\}$ with $i_1<\cdots<i_{n-2}$.

For an arbitrary integer $0\leqq s\leqq n$ we consider a C^∞ s-form β on U whose local expression in x is

$$\beta=\sum_{i_1<\cdots<i_s} b_{i_1\cdots i_s}(x)dx^{i_1}\wedge\cdots\wedge dx^{i_s}.$$

Besides the original point norm $|\beta|$ of β we consider the point norm $|\beta|_e$ with respect to the flat metric tensor (δ_{ij}) on U:

$$|\beta|_e^2 = \sum_{i_1<\cdots<i_s}\Big(\sum_{j_1<\cdots<j_s}\delta^{i_1j_1}\cdots\delta^{i_sj_s}b_{j_1\cdots j_s}\Big)b_{i_1\cdots i_s} = \sum_{i_1<\cdots<i_s}(b_{i_1\cdots i_s})^2.$$

Here we note that $|\beta|_e(\xi)=|\beta|(\xi)$ since $g^{ij}(0)=\delta^{ij}$. Now let $\lambda_1(x),\ldots,\lambda_n(x)$ be the proper values of the matrix $(g^{ij}(x))$. Then the proper values of the tensor product $(g^{ij})\otimes\cdots\otimes(g^{ij})$ of s same matrices (g^{ij}) are $\lambda_{i_1}\cdots\lambda_{i_s}(i_1,\ldots,i_s=1,\ldots,n)$. Since the second part of (8) is equivalent to $K^{-1}\leqq\lambda_i\leqq K$ $(i=1,\ldots,n)$ on B_δ, we have $K^{-s}\leqq\lambda_{i_1}\cdots\lambda_{i_s}\leqq K^s$ $(i_1,\ldots,i_s=1,\ldots,n)$ on B_δ, which is equivalent to

$$K^{-s}(\delta^{ij})\otimes\cdots\otimes(\delta^{ij})\leqq(g^{ij})\otimes\cdots\otimes(g^{ij})\leqq K^s(\delta^{ij})\otimes\cdots\otimes(\delta^{ij})$$

on B_δ. This implies the following inequalities valid on U_δ:

(10) $K^{-s}|\beta|_e^2\leqq|\beta|^2\leqq K^s|\beta|_e^2$.

Take any $\varphi\in C_0^\infty(U_\delta)\setminus\{0\}\,(\subset C_0^\infty(M)\setminus\{0\})$. Recall that $dV=\sqrt{g}\,dx$ where $g=\det(g_{ij})$ and $dx=dx^1\wedge\cdots\wedge dx^n$ on U. The first part of (8) implies that $\det(K^{-1}(\delta_{ij}))\leqq\det(g_{ij})\leqq\det(K(\delta_{ij}))$ so that $K^{-n/2}\leqq\sqrt{g}\leqq K^{n/2}$ on U_δ. Using (10) for α, $d\varphi$ and $\alpha\wedge d\varphi$ we see that

$$\int_{U_\delta}|\alpha\wedge d\varphi|_e^2dx\leqq K^{(n-1)+n/2}\int_{U_\delta}|\alpha\wedge d\varphi|^2\sqrt{g}dx$$
$$=K^{3n/2-1}\|\alpha\wedge d\varphi\|_2^2\leqq K^{3n/2-1}N[\alpha]^2\|d\varphi\|_2^2$$
$$=K^{3n/2-1}N[\alpha]^2\int_{U_\delta}|d\varphi|^2\sqrt{g}dx$$
$$\leqq K^{3n/2-1}N[\alpha]^2K^{1+n/2}\int_{U_\delta}|d\varphi|_e^2dx$$

so that on noting $\varphi\in C_0^\infty(U_\delta)$ we have obtained

(11) $\int_U|\alpha\wedge d\varphi|_e^2dx\leqq K^{2n}N[\alpha]^2\int_U|d\varphi|_e^2dx$.

We now specialize $\varphi\in C_0^\infty(U_\delta)\setminus\{0\}$. For the purpose take a small $\tau>0$ such that the interval $[-\tau,\tau]\times\cdots\times[-\tau,\tau]$ (n factors) is contained in U_δ. We can find a $\psi\in C_0^\infty(\boldsymbol{R}^1)$ such that supp $\psi\subset[-\tau,\tau]$ and $A:=\int_{-\tau}^\tau\psi(t)^2dt>0$ so that $B:=\int_{-\tau}^\tau\psi'(t)^2dt>0$. On setting $\varphi(x)=\Pi_{i=1}^n\psi(x^i)$ we may view that $\varphi\in C_0^\infty(U_\delta)\setminus\{0\}\subset C_0^\infty(M)\setminus\{0\}$. Then we see that $D_i\varphi(x)=\psi'(x^i)\Pi_{j\neq i}\psi(x^j)$ $(i=1,\ldots,n)$

and

$$|d\varphi(x)|_e^2 = \sum_{i=1}^n (D_i\varphi(x))^2 = \sum_{i=1}^n \phi'(x^i)^2 \prod_{j\neq i} \phi(x^j)^2.$$

By using the Fubini theorem we see that $\int_U (D_i\varphi(x))^2 dx$ equals

$$\int_U \phi'(x^i)2 \prod_{j\neq i} \phi(x^j)^2 dx = \left(\int_{-\tau}^{\tau} \phi'(x^i)^2 dx^i\right) \prod_{j\neq i} \int_{-\tau}^{\tau} \phi(x^j)^2 dx^j = A^{n-1}B$$

for every $i = 1, \ldots, n$. Thus on setting $C := A^{n-1}B$ we obtain

(12) $$\int_U (D_i\varphi(x))^2 dx = C \ (i = 1, \ldots, n) \text{ and } \int_U |d\varphi(x)|_e^2 dx = nC.$$

From (7) and (9) we infer that

$$|\alpha \wedge d\varphi|_e^2 = \sum_{j=1}^n \Big(\sum_{i_1<\cdots<i_{n-2}}^{j} a_{i_1\cdots i_{n-2}}(x)^2 (D_j\varphi(x))^2 \Big) \geqq \sum_{j=1}^n \Big(\sum_{i_1<\cdots<i_{n-2}}^{j} (a_{i_1\cdots i_{n-2}}(0)^2 - \varepsilon)(D_j\varphi(x))^2 \Big)$$

on U_δ and hence on U by $\varphi \in C_0^\infty(U_\delta)$. Therefore we obtain

$$\sum_{j=1}^n \Big(\sum_{i_1<\cdots<i_{n-2}}^{j} (a_{i_1\cdots i_{n-2}}(0)^2 - \varepsilon) \int_U (D_j\varphi(x))^2 dx \Big) \leqq \int_U |\alpha \wedge d\varphi|_e^2 dx.$$

From this with (11) and (12) we deduce that

$$\sum_{j=1}^n \Big(\sum_{i_1<\cdots<i_{n-2}}^{j} (a_{i_1\cdots i_{n-2}}(0)^2 - \varepsilon) \Big) \leqq nK^{2n} N[\alpha]^2.$$

Since $K > 1$ and $\varepsilon > 0$ were chosen arbitrarily, we may let $K \downarrow 1$ and $\varepsilon \downarrow 0$ in the above inequality to conclude that

$$\sum_{j=1}^n \Big(\sum_{i_1<\cdots<i_{n-2}}^{j} a_{i_1\cdots i_{n-2}}(0)^2 \Big) \leqq nN[\alpha]^2.$$

The left hand side term of the above inequality can be easily seen to coincide with $2|\alpha|(\xi)^2$. Here we have used $|\alpha|(\xi) = |\alpha|_e(\xi)$. Therefore $|\alpha|(\xi)^2 \leqq (n/2)N[\alpha]^2$. Since $\xi \in M$ is arbitrary, we finally conclude that $\|\alpha\|_\infty^2 \leqq (n/2)N[\alpha]^2$. □

We let M be the Euclidean n-space $\boldsymbol{R}^n$ with the usual Euclidean metric tensor (δ_{ij}) in the following two examples. First, if we take $\alpha = \sum_{i_1<\cdots<i_{n-2}} dx^{i_1} \wedge \cdots \wedge dx^{i_{n-2}}$ in the natural coordinate $x = (x^1, \ldots, x^n)$ on $\boldsymbol{R}^n$, then $N[\alpha] = (n/2)^{-1/2}\|\alpha\|_\infty$. In fact, for any $\varphi \in C_0^\infty(\boldsymbol{R}^n)\backslash\{0\}$, we see that $|\alpha \wedge d\varphi|^2 = (n-1)|d\varphi|^2$ and thus $N[\alpha] \leqq (n-1)^{1/2}$. Since $|\alpha|^2 = n(n-1)/2$, $\|\alpha\|_\infty = (n(n-1)/2)^{1/2}$. A fortiori, $N[\alpha] \leqq (n/2)^{-1/2}\|\alpha\|_\infty$, which with the left hand side of (5) implies that $N[\alpha] = (n/2)^{-1/2}\|\alpha\|_\infty > 0$. Next, we show the existence of a C^∞ $(n-2)$-form α on $\boldsymbol{R}^n$ such that $N[\alpha] = \|\alpha\|_\infty > 0$. Since this is always the case for any C^∞ $(2-2)$-form $\alpha \neq 0$ on $\boldsymbol{R}^2$, we only have to consider the case $n \geqq 3$. Then the required α is $\alpha = dx^3 \wedge dx^4 \wedge \cdots \wedge dx^n$ in the natural coordinate $x = (x^1, \ldots, x^n)$ on $\boldsymbol{R}^n$. In fact, we can choose a $\psi_m \in C_0^\infty(\boldsymbol{R}^1)$ such that $\int_{-\infty}^{\infty} \psi_m'(t)^2 dt = m$ and $\int_{-\infty}^{\infty} \psi_m(t)^2 dt = \mathcal{O}(1/m)$ for each $m = 1, 2, \ldots$. We also choose a $\phi \in C_0^\infty(\boldsymbol{R}^1)$ such that $A := \int_{-\infty}^{\infty} \phi(t)^2 dt > 0$ and $B := \int_{-\infty}^{\infty} \phi'(t)^2 dt > 0$. Then set $\varphi_m(x) = \psi_m(x^1)\Pi_{j=2}^n \phi(x^j)$, which belongs to $C_0^\infty(\boldsymbol{R}^n)$. We have

$$\|d\varphi_m\|_2^2 = mA^{n-1} + \mathcal{O}(1/m)\cdot(n-1)A^{n-2}B,$$
$$\|\alpha \wedge d\varphi\|_2^2 = mA^{n-1} + \mathcal{O}(1/m)\cdot A^{n-2}B.$$

Thus $N[\alpha]^2 \geqq \sup_m \|\alpha \wedge d\varphi_m\|_2^2 / \|d\varphi_m\|_2^2 = 1 = \|\alpha\|_\infty^2$ so that $N[\alpha] \geqq \|\alpha\|_\infty$, which with the right hand side of (5) implies that $N[\alpha] = \|\alpha\|_\infty$.

Finally we consider the case M is a subregion of $\boldsymbol{R}^n$ which is homeomorphic to the unit ball $\{x \in \boldsymbol{R}^n : |x| < 1\}$. The proofs of the following facts will appear elsewhere.

Fact 1. *If the boundary ∂M of M relative to $\boldsymbol{R}^n$ is a smooth closed hypersurface, then M possesses the Virtanen property.*

Fact 2. *If $M = \{x \in \boldsymbol{R}^n : 1/2 < |x| < 1\} \backslash \{(x^1, 0, \ldots, 0) \in \boldsymbol{R}^n : 1/2 < x^1 < 1\}$, then M does not possess the Virtanen property.*

Fact 3. *If M is star-shaped with respect to a point in M, then M possesses the Virtanen property.*

Every point in ∂M in Fact 1 is Dirichlet-regular. The M in Fact 2 contains the set $\{(x^1, 0, \ldots, 0) \in \boldsymbol{R}^n : 1/2 < x^1 < 1\}$ of irregular points in its boundary. Looking at these two results one might feel that the existence of irregular points in ∂M has something to do with the Virtanen property for M. From this view point the significance of Fact 3 lies in the fact that there is a star-shaped region M whose boundary contains the Lebesgue spine (cf. e.g. [2]), an irregular boundary point.

References

[1] G. de Rham: Variétés Différentiables. Hermann (1995).

[2] L. L. Helms: Introduction to Potential Theory.

Wiley-Interscience (1969).
[3] J. Heinonen *et al.*: Nonlinear Potential Theory of Degenerate Elliptic Equations. Oxford University Press (1993).
[4] M. Nakai: Riemannian manifolds with connected Royden harmonic boundaries. Duke Math. J., **67**, 589–625 (1992).