A Note on the Capitulation in Z_p-extensions

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1. Introduction. Let K be a number field, namely, a finite extension field over the field of rational numbers. Let p be a prime number, and K_∞/K a $\boldsymbol{Z}_p$-extension with Galois group Γ. Denote by K_n the n-th layer of K_∞/K. We write A_n for the p-Sylow subgroup of the ideal class group of K_n, and $j_{n,m}$ for the map from A_n to A_m induced from the inclusion $K_n \subseteq K_m$ for $m \geq n \geq 0$. Let $A_\infty = \varinjlim A_n$, where the inductive limit is taken with respect to $j_{n,m}$, and let $j_{n,\infty}$ be the natural map from A_n to A_∞. We denote by $\lambda(K_\infty/K)$, $\mu(K_\infty/K)$ the Iwasawa λ, μ invariant, respectively, of K_∞/K.

In the present paper, we shall prove the following:

Theorem. *Let notations be as above. If we assume that all primes of K which ramify in K_∞ are totally ramified in K_∞, then*

$$\lambda(K_\infty/K) = \mu(K_\infty/K) = 0 \Leftrightarrow Ker(N_{1,0}: A_1 \to A_0) \subseteq Ker(j_{1,\infty}: A_1 \to A_\infty),$$

where $N_{1,0}$ stands for the norm map from A_1 to A_0.

This theorem is in analogy to the following theorem due to Greenberg [1, Theorem 1], though our proof is based on a method different from [1].

Theorem (Greenberg). *Let K be a totally real number field, and K_∞/K a $\boldsymbol{Z}_p$-extension. If we assume that there is only one prime in K above p, which is totally ramified in K_∞, then*

$$\lambda(K_\infty/K) = \mu(K_\infty/K) = 0 \Leftrightarrow Ker(j_{0,\infty}: A_0 \to A_\infty) = A_0.$$

2. Proof of Theorem. We fix a topological generator γ of Γ. Put $\Lambda = \boldsymbol{Z}_p[[\Gamma]]$ and $\nu_{n,m} = \dfrac{\gamma^{p^m} - 1}{\gamma^{p^n} - 1} \in \Lambda$ for $m \geq n \geq 0$. Let L_∞/K_∞ be the maximal unramified pro-p abelian extension, and let X be its Galois group. Put $Y = Gal(L_\infty/K_\infty L_0)$, where L_0 is the Hilbert p-class field of K. Then X is a finitely generated torsion Λ-module, and the Artin map induces the isomorphism $A_n \simeq X/\nu_{0,n}Y$ for all $n \geq 0$ (cf. [2. Theorem 6]). So we will identify $X/\nu_{0,n}Y$ with A_n.

We first prove the following:

Proposition. $Ker(j_{n,\infty}: A_n \to A_\infty) = Im(X_{finite} \to X/\nu_{0,n}Y)$ *for all $n \geq 0$, where X_{finite} denotes the maximal finite Λ-submodule of X.*

Proof. Since X_{finite} is finite, we see that $\nu_{n,m}X_{finite} = 0$ for some $m \geq n$. Observing the following commutative diagram (cf. [2, Theorem 7]):

$$(1)\qquad \begin{array}{ccc} A_m & \xrightarrow{\sim} & X/\nu_{0,m}Y \\ {\scriptstyle j_{n,m}}\uparrow & & \uparrow{\scriptstyle \nu_{n,m}} \\ A_n & \xrightarrow{\sim} & X/\nu_{0,n}Y, \end{array}$$

we have

$$Im(X_{finite} \to X/\nu_{0,n}Y) \subseteq Ker(j_{n,\infty}: A_n \to A_\infty).$$

Conversely, let $x\, mod.\nu_{0,n}Y \in X/\nu_{0,n}Y$ be any element in $Ker(j_{n,\infty}: A_n \to A_\infty)$. It follows from (1) that $\nu_{n,m}x \in \nu_{0,m}Y$ for some $m \geq n$. Hence $\nu_{n,m}x = \nu_{0,m}y$ for some $y \in Y$, that is

$$(2)\qquad \nu_{n,m}(x - \nu_{0,n}y) = 0.$$

X/X_{finite} is embedded in the Λ-module $\bigoplus_{i=1}^r \Lambda/f_i^{e_i}\Lambda$ with finite cokernel, where $f_i \in \Lambda$ is a prime element. Since $X/\nu_{n,m}X$ is finite, $\prod_{i=1}^r f_i^{e_i}$ is prime to $\nu_{n,m}$. So we see that the multiplication-by-$\nu_{n,m}$ map $\nu_{n,m}: X/X_{finite} \to X/X_{finite}$ is injective. Therefore we have $x - \nu_{0,n}y \in X_{finite}$ from (2). So we obtain $x\, mod.\nu_{0,n}Y \in Im(X_{finite} \to X/\nu_{0,n}Y)$. □

Corollary. $Ker(j_{n,\infty}: A_n \to A_\infty) \neq 0$ for some $n \geq 0 \Leftrightarrow X_{finite} \neq 0$

Proof. ($\Rightarrow$) part is obvious by Proposition. We assume that $X_{finite} \neq 0$. It follows from $\bigcap_{n\geq 0} \nu_{0,n}Y = 0$ that $X_{finite} \not\subseteq \nu_{0,n}Y$ for some $n \geq 0$. Therefore we find that $Ker(j_{n,\infty}: A_n \to A_\infty) \neq 0$ by Proposition. □

Proof of Theorem. ($\Rightarrow$) part is easy (cf. [1, Proposition 2]). We assume that $Ker(N_{1,0}: A_1 \to A_0) \subseteq Ker(j_{1,\infty}: A_1 \to A_\infty)$. From the commutative diagram (cf. [2, Theorem 7])

$$(3)\qquad \begin{array}{ccc} A_1 & \xrightarrow{\sim} & X/\nu_{0,1}Y \\ {\scriptstyle N_{1,0}}\downarrow & & \downarrow{\scriptstyle projection} \\ A_0 & \xrightarrow{\sim} & X/Y, \end{array}$$

we see that $Ker(N_{1,0}: A_1 \to A_0) = Y/\nu_{0,1}Y$. It follows from the assumption and Proposition that

$Y = Y \cap X_{finite} + \nu_{0,1} Y$. Since $\nu_{0,1}$ is contained in the maximal ideal of Λ, we obtain that $Y = Y \cap X_{finite}$ by Nakayama's lemma. Therefore Y is finite. Hence the finiteness of $X/Y = A_0$ implies that X is also finite. Therefore $\lambda(K_\infty/K) = \mu(K_\infty/K) = 0$. □

References

[1] R. Greenberg: On the Iwasawa invariants of totally real number fields. Amer. J. of Math., **98**, 263–284 (1976).

[2] K. Iwasawa: On $\boldsymbol{Z}_l$-extensions of algebraic number fields. Ann. of Math., **98**, 246–326 (1973).