Cubic Hyper-equisingular Families of Complex Projective Varieties. I

By Shoji TSUBOI

Department of Mathematics, Kagoshima University

(Communicated by Heisuke HIRONAKA, M. J. A., Nov. 13, 1995)

Introduction. The purpose of this note is to outline a recent result of the author's study on *cubic hyper-equisingular families of complex projective varieties,* from which there naturally arise variations of mixed Hodge structure. In order to define such families we use *cubic hyper-resolutions* of complex projective varieties due to V. Navarro Aznar, F. Guillén *et al.*, [1]. The initial motivation for this study was to describe the variation of mixed Hodge structure which might be expected to arise from a *locally trivial* family of projective varieties with *ordinary singularities* (cf. [3], [4]). Details will be published elsewhere.

§1. Cubic hyper-equisingular families of complex projective varieties. We denote by $\boldsymbol{Z}$ the integer ring.

1.1 Definition. For $n \in \boldsymbol{Z}$ with $n \geq 0$ the *augmented n-cubic category,* denoted by $\square_n^+$, is defined to be a category whose objects $\mathrm{Ob}(\square_n^+)$ and the set of homomorphisms $\mathrm{Hom}_{\square_n^+}(\alpha, \beta)$ ($\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_0, \beta_1, \ldots, \beta_n) \in \mathrm{Ob}(\square_n^+)$) are given as follows:

$$\mathrm{Ob}(\square_n^+) := \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \boldsymbol{Z}^{n+1} \mid 0 \leq \alpha_i \leq 1 \text{ for } 0 \leq i \leq n\},$$

$$\mathrm{Hom}_{\square_n^+}(\alpha, \beta) := \begin{cases} \alpha \to \beta \text{ (an arrow from } \alpha \text{ to } \beta) & \text{if } \alpha_i \leq \beta_i \text{ for } 0 \leq i \leq n \\ \emptyset & \text{otherwise.} \end{cases}$$

For $n = -1$ we understand $\square_{-1}^+$ to be the punctual category $\{*\}$, i. e., the category consisting of one point. For $n \geq 0$ the *n-cubic category,* denoted by $\square_n$, is defined to be the full subcategory of $\square_n^+$ with $\mathrm{Ob}(\square_n) = \mathrm{Ob}(\square_n^+) - \{(0, \ldots, 0)\}$. Notice that $\mathrm{Ob}(\square_n^+) - \{(0, \ldots, 0)\}$ (resp. $\mathrm{Ob}(\square_n)$) can be considered as a finite ordered set whose order is defined by $\alpha \leq \beta \Leftrightarrow \alpha \to \beta$ for $\alpha, \beta \in \mathrm{Ob}(\square_n^+)$ (resp. $\mathrm{Ob}(\square_n)$).

1.2 Definition. A $\square_n^+$-object (resp. $\square_n$-object) of a category $\mathscr{C}$ is a contravariant functor $X_\cdot^+$ (resp. $X.$) from $\square_n^+$ (resp. $\square_n$) to $\mathscr{C}$. It is also called an *augmented n-cubic object of* $\mathscr{C}$ (resp. an *n-cubic object of* $\mathscr{C}$).

1.3 Definition. Let $X.$, $Y.$ be $\square_n^+$-objects of a category $\mathscr{C}$. We define a morphism $\Phi.: X. \to Y.$ to be a natural transformation from the functor $X.$ to the one $Y.$ over the identity functor id: $\square_n^+ \to \square_n^+$.

Let $X.$ be an n-cubic object of $\mathscr{C}$ ($n \geq 0$), X a (-1)-object of $\mathscr{C}$. We denote by $X \times \square_n$ the n-cubic object defined by $(X \times \square_n)(\alpha) = X$ for every $\alpha \in \square_n$. An *augmentation of* $X.$ *to* X is a morphism from $X.$ to $X \times \square_n$. We may think of an n-cubic object of $\mathscr{C}$ with an augmentation to X as an augmented n-cubic object of $\mathscr{C}$. Conversely, an augmented n-cubic object $X_\cdot^+: (\square_n^+)^\circ \to \mathscr{C}$ of $\mathscr{C}$ can be identified with an n-cubic object $X. := X_{\cdot|\square_n}^+: (\square_n)^\circ \to \mathscr{C}$ of $\mathscr{C}$ with an augmentation to $X_{(0,\ldots,0)}^+$. In the following we shall interchangeably use an augmented n-cubic object of $\mathscr{C}$ and an n-cubic object of $\mathscr{C}$ with an augmentation.

1.4 Definition. For a $\square_n^+$-complex projective variety $X.$, a contravariant functor $Y.$ from $\square_1^+$ to the category of $\square_n^+$-complex projective varieties is called a 2-*resolution of* $X.$ if $Y.$ is defined by a cartesian square of morphisms of $\square_n^+$-complex projective varieties

$$(1.1)\qquad \begin{array}{ccc} Y_{11}. & \longrightarrow & Y_{01}. \\ \downarrow & & \downarrow f \\ Y_{10}. & \longrightarrow & Y_{00}., \end{array}$$

which satisfies the following conditions:

(i) $Y_{00}. = X.$,

(ii) $Y_{01}.$ is a smooth $\square_n^+$-complex projective variety, i.e., a contravariant functor from $\square_n^+$ to the category of smooth complex projective varieties,

(iii) the horizontal arrows are closed immersion of $\square_n^+$-complex projective varieties,

(iv) f is a proper morphism between $\square_n^+$-complex projective varieties, and

(v) f induces an isomorphism from $Y_{01\beta} - Y_{11\beta}$ to $Y_{00\beta} - Y_{10\beta}$ for any $\beta \in \mathrm{Ob}(\square_n^+)$.

We think of the cartesian square in (1.1) as a morphism from the $\square_{n+1}^+$-complex projective variety $Y_{1\cdot\cdot}$ to the one $Y_{0\cdot\cdot}$ and write it as $Y_{1\cdot\cdot} \to Y_{0\cdot\cdot}$. For a 2-resolution $Z.$ of $Y_{1\cdot\cdot}$, we define the

$\square^+_{n+3}$-complex projective variety $rd(Y., Z.)$ by

$$rd(Y., Z.) := \begin{array}{ccc} Z_{11}. & \longrightarrow & Z_{01}. \\ \downarrow & & \downarrow \\ Z_{10}. & \longrightarrow & Y_{0}.., \end{array}$$

and call it the *reduction* of $\{Y., Z.\}$.

1.5 Definition. Let X be a complex projective variety and let $\{X^1_., X^2_., \cdots, X^n_.\}$ be a sequence of $\square^+_r$-complex projective varieties $X^r_.(1 \leq r \leq n)$ such that

(i) $X^1_.$ is a two resolution of X, and

(ii) $X^{r+1}_.$ is a two resolution of $X^r_1.$ for $1 \leq r \leq n-1$.

Then, by induction on n, we define

$$Z. := rd(X^1_., X^2_., \cdots, X^n_.)$$
$$:= rd(rd(X^1_., X^2_., \cdots, X^{n-1}_.), X^n_.).$$

With this notation, if Z_α are smooth for all $\alpha \in \square_n$, we call $Z.$ an *augmented n-cubic hyper-resolution of* X.

We denote by $\mathscr{F}_M(\mathrm{Proj}/C)$ (resp. $\mathscr{F}_M(\mathrm{An}/C)$) the category of analytic famities of complex projective (resp. analytic) varieties, parametrized by a complex space M.

1.6 Definition. we call a $\square^+_n$-object (resp. $\square_n$-object) of $\mathscr{F}_M(\mathrm{Proj}/C)$ (resp. $\mathscr{F}_M(\mathrm{An}/C)$) *an analytic family of augmented n-cubic* (resp. *n-cubic*) *complex projective* (*resp. analytic*) *varieties, parametrized by a complex space* M.

Let $b.: X. \to X$ be an augmented n-cubic complex projective (resp. analytic) variety and M a complex space. Then $X_\alpha \times M$ ($\alpha \in \square_n$), $X \times M$, $a_\alpha := b_\alpha \times \mathrm{id}_M : X_\alpha \times M \to X \times M$ and $\pi := \mathrm{Pr}_M : X \times M \to M$, the projection to M, constitute an analylie family of augmented n-cubic complex projective (resp. analytic) varieties, parametrized by a complex space M, which we denote by

$$X. \times M \xrightarrow{a.:=b.\times \mathrm{id}_M} X \times M \xrightarrow{\pi:=\mathrm{Pr}_M} M$$

and call the *product family of augmented n-cubic complex projective* (*resp. analytic*) *varieties, parametrized by a complex space* M. Let $\mathscr{X}^+_. = \{a.: \mathscr{X}. \to \mathscr{X}\}$ be an analytic family of augmented n-cubic complex projective (resp. analytic) varieties, parametrized by a complex space M. Whenever we wish to express its parameter space M explicitly, we write

$$\mathscr{X}. \xrightarrow{a.} \mathscr{X} \xrightarrow{\pi} M. \tag{1.2}$$

For $t \in M$, $X_{t\alpha} := (\pi \cdot a_\alpha)^{-1}(t)$ ($\alpha \in \square_n$), $X_t := \pi^{-1}(t)$ and $a_{t\alpha} := a_{\alpha|X_{t\alpha}} : X_{t\alpha} \to X_t$ constitute an augmented n-cubic complex projective (resp. analytic) variely, which we denote by $a_t : X_{t.} \to X_t$ and call the *fiber at* $t \in M$ of an analytic family of augmented n-cubic complex projective (resp. analytic) varietics in (1.2). Similarly, for an open subset U of $\mathscr{X}$, we form an analytic family

$$a.^{-1}(\mathscr{U}) \xrightarrow{a.|a.^{-1}(\mathscr{U})} \mathscr{U} \xrightarrow{\pi} \pi(\mathscr{U})$$

of augmented n-cubic analytic varieties, parametized by a complex space $\pi(\mathscr{U})$. With these notions, we define an n-cubic hyper-equisingular family of complex projective varieties, parametrized by a complex space as fallows:

1.7 Definition. Let $\mathscr{X}. \xrightarrow{a.} \mathscr{X} \xrightarrow{\pi} M$ be a family of augmented n-cubic complex projective varieties, parametrized by a complex space M. We call $\mathscr{X}. \xrightarrow{a.} \mathscr{X} \xrightarrow{\pi} M$ an *n-cubic hyper-equisingular family of complex projective varieties, parametrized by a complex space* M if it satisfies the fallowing conditions:

(i) for any point $t \in M$, $a_t.: X_t. \to X_t$ is an augmented n-cubic hyper-resolution of X_t,

(ii) (analytical "local triviality") for any point $p \in \mathscr{X}$, there exists an open neighborhood $\mathscr{U}$ of p in $\mathscr{X}$ such that $a.^{-1}(\mathscr{U}) \xrightarrow{a.} \mathscr{U} \xrightarrow{\pi} \pi(\mathscr{U})$ is analytically isomorphic to

$$(a.^{-1}(\mathscr{U}) \cap X_{\pi(p)}.) \times \pi(\mathscr{U}) \to (\mathscr{U} \cap X_{\pi(p)}) \times \pi(\mathscr{U}) \xrightarrow{\mathrm{Pr}_{\pi(\mathscr{U})}} \pi(\mathscr{U})$$

over the identy map $\mathrm{id}_{\pi(\mathscr{U})} : \pi(\mathscr{U}) \to \pi(\mathscr{U})$.

1.8 Proposition. *Let* $\mathscr{X}. \xrightarrow{a.} \mathscr{X} \xrightarrow{\pi} M$ *be an n-cubic hyper-equisingular family of complex projective varieties, parametrized by a complex manifold* M. *Then the* $\square_n$*-object* $\pi.: \mathscr{X}. \to M$ ($\pi. := \pi \circ a.$) *of smooth families of complex manifolds, parametrized by* M *is* C^∞ *trivial at any point of* M; *that is, for any point* $t_0 \in M$, *there exist an open neighborhood* N *of* t_0 *in* M *and a diffeomorphism* $\Phi.: (\pi.^{-1})(N) \to X_{t_0}. \times N$ *of* $\square_n$*-objects of complex manifolds over the identy map* $\mathrm{id}_N : N \to N$. *Furthermore,* $\mathscr{X}. \xrightarrow{a.} \mathscr{X} \xrightarrow{\pi} M$ *is topologically trivial at any point of* M.

1.9 Example. By a *locally trivial family of complex projective varieties, parametrized by a complex space* M, we mean an analytic family $\pi: \mathscr{Z} \to M$ of complex projective varieties, parametrized by a complex space M, which satisfies the following condition: for every point $p \in \mathscr{Z}$, there exist open neighborhoods $\mathscr{U}$ of p in $\mathscr{Z}$, V of $\pi(p)$ in M with $\pi(\mathscr{U}) = V$, and a biholomorphic map $\phi: \mathscr{U}$

$\longrightarrow U \times V$, where we define $U := \mathcal{U} \cap Z_{\pi(p)}$, such that (a) $\mathrm{Pr}_V \circ \phi = \pi_{|\mathcal{U}}$ and (b) $\phi_{|U\times p} := \mathrm{id}_{U\times p}$. With this notion, we get cubic hyper-equisingular families of complex projective varieties from locally trivial families of complex projective varieties with *ordinary singularities* of dimension ≤ 5 as well as from locally trivial families of complex projective varieties with *normal crossing* of any dimension by taking their *simultaneous* cubic hyper-resolutions. Here we say that a complex projective variety is *with ordinary singularities* if it is locally isomorphic to one of the germs of pure-dimensional hypersurfaces with locally stable parametrizations in a complex manifold (cf. [3], [4]).

§2. Cohomological descent for cubic hyper-equisingular families. The relative version of "cohomological descent" holds for a cubic hyper-equisingular family of complex projective varieties. In order to state these facts we prepare some notation and terminology. Let $\Phi.: X. \longrightarrow X$ be an n-cubic topological space with an augmentation to a topological space X. We denote by $\mathcal{M}(X., R)$ and $\mathcal{M}(X, R)$ the categories of R-module sheaves on $X.$ and X, respectively, where R is a commutative ring. For an R-module sheaf $\mathcal{F}$ on X we define its inverse image $\Phi.^*\mathcal{F} \in \mathcal{M}(X., R)$ in a natural way. The functor $\Phi.^*: \mathcal{M}(X, R) \longrightarrow \mathcal{M}(X., R)$ has a right adjoint $\Phi._*: \mathcal{M}(X., R) \longrightarrow \mathcal{M}(X, R)$. Since the functor $\Phi.^*$ is exact, it defines a functor

(2.1) $\qquad \Phi.^*: D^+(X, R) \longrightarrow D^+(X., R),$

where $D^+(X, R)$ and $D^+(X., R)$ denote the derived categories of lower bounded complexes of R-module sheaves on X and $X.$, respectively. The functor in (2.1) has a right adjoint

$$R\Phi._*: D^+(X., R) \longrightarrow D^+(X, R).$$

For more details we refer to [1, Exposé I].

2.1 Theorem (*Cohomological descent of R-module sheaves*). *Let $\mathcal{X}. \xrightarrow{a.} \mathcal{X} \xrightarrow{\pi} M$ be an n-cubic ($n \geq 1$) hyper-equisingular family of complex projective varieties, parametrized by a complex space M. Then, for an R-module sheaf $\mathcal{A}$ on $\mathcal{X}$, the adjunction map*

$$\mathcal{A} \longrightarrow Ra._*a.^*\mathcal{A}$$

is an isomorphism in $D^+(\mathcal{X}, R)$.

For an n-cubic hyper-equisingular family $\mathcal{X}. \xrightarrow{a.} \mathcal{X} \xrightarrow{\pi} M$ of complex projective varieties, parametrized by a complex space M, we denote by $\Omega_{\mathcal{X}_\alpha/M}$ the relative de Rham complex of a smooth family $\pi \circ a_\alpha : \mathcal{X}_\alpha \longrightarrow M$ of complex manifolds for each $\alpha \in \square_n$. Then $\Omega^\cdot_{\mathcal{X}./M} := \{\Omega^\cdot_{\mathcal{X}_\alpha/M}\}_{\alpha\in\square_n}$ is obviously a complex of sheaves of C-vector spaces on a $\square_n$-complex manifold $\mathcal{X}.$.

2.2 Theorem (*Cohomological descent of relative de Rham complexes*). *Under the same setting as above, there naturally exists an isomorphism*

$$DR^\cdot_{\mathcal{X}/M} \simeq Ra._*\Omega^\cdot_{\mathcal{X}./M}$$

in $D^+(\mathcal{X}, C)$, where $DR^\cdot_{\mathcal{X}/M}$ is the cohomological relative de Rham complex of a locally trivial family $\pi: \mathcal{X} \longrightarrow M$, i.e., the relative version of a cohomological de Rham complex of a singular variety (*cf.* [2, *p.*28, *Remark*]).

The proofs of these theorems are almost identical with those in the absolute cases, i.e., M is a single point (cf. [1, p.41, Théorèm 6.9], [1, p.61, Théorèm 1.3]).

References

[1] F. Guillén, V. Navarro Aznar, P. Pascual-Gainza and F. Puerta: Hyperrésolutions cubiques et descente cohomologique. Lect. Notes in Math., vol. 1335, Springer, Berlin (1988).

[2] R. Hartshorne: On the de Rham cohomology of algebraic varieties. Publ. Math. IHES, **45**, 5–99 (1976).

[3] S. Tsuboi: Deformations of locally stable holomorphic maps and locally trivial displacements of analytic subvarieties with ordinary singularities. Sci. Rep. Kagoshima Univ., **35**, 9–90 (1986).

[4] S. Tsuboi: Global existence of the universal locally trivial family of analytic subvarieties with locally stable parametrizations of a compact complex manifold. J. Fac. Sci. Univ. Tokyo, **40**, no.1, 161–201 (1993).

[5] S. Tsuboi: Variations of mixed Hodge structure arising from cubic hyperequisingular families of complex projective varieties. I. II (Preprint series no.22, no.23, Institute of Mathematics, University of Oslo, 1995).