On Power Series Attached to Local Densities

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Let p be a prime number different form 2. For non-degenerate symmetric matrices S and T of degrees s and $t(s \geq t \geq 1)$, respectively, with entries in the ring $\boldsymbol{Z}_p$ of p-adic integers, we define the local density $\alpha_p(T, S)$ by

$$\alpha_p(T, S) = \lim_{e \to \infty} \# \mathscr{A}_e(T, S),$$

where

$$\mathscr{A}_e(T, S) = \{X \in M_{mn}(\boldsymbol{Z}_p)/p^e M_{mn}(\boldsymbol{Z}_p) ; S[X] - T \in p^e M_{mn}(\boldsymbol{Z}_p)\}.$$

For the precise definition, see [2]. In [2] we defined a formal power series $P(T, S; x_1, \ldots, x_t)$ by

$$P(T, S; x_1, \ldots, x_t) = \sum_{r_1, \ldots, r_t = 0}^{\infty} \alpha_p(T[\mathrm{diag}(p^{r_1}, \ldots, p^{r_t})], S) x_1^{r_1} \ldots x_t^{r_t}.$$

In [3] we showed that $P(T, S; x_1, \ldots, x_t)$ is a rational function of $x_1, \ldots, x_t$ over the field $\boldsymbol{Q}$ of rational numbers for arbitrary S and T. In [4] we gave an explicit form of its denominator for the case where T is a diagonal matrix. In view of the theory of Siegel Eisenstein series, it is important to give a precise information on its denominator and numerator for the case where $S = \frac{1}{2}\begin{pmatrix} 0 & E_k \\ E_k & 0 \end{pmatrix}$ with $k > t + 1$ (cf. Kitaoka [7]).

In the present paper, we give a more precise form of its denominator and the degree of its numerator for the case where S is a unimodular matrix of degree not smaller than $2 \deg T$. To state our first main result, for integers n, β, γ, put

$$\Delta(n, \beta, \gamma) = \{\{i_1, \ldots, i_\beta, j_1, \ldots, j_\gamma\} ; 1 \leq i_1 < \ldots < i_\beta \leq n, 1 \leq j_1 < \ldots < j_\gamma \leq n, \{i_1, \ldots, i_\beta\} \cap \{j_1, \ldots, j_\gamma\} = \emptyset\}.$$

Then we have

Theorem 1. *Assume that $s \geq 2t$, and that T is a diagonal matrix. Then the denominator of $P(T, S; x_1, \ldots, x_t)$ is of the following form:*

$$\prod_{\beta=1}^{t} \prod_{\gamma=1}^{t-\beta} \prod_{\{i_1, \ldots, i_\beta, j_1, \ldots, j_\gamma\}} (1 - p^{\beta(-s+t+\gamma+1)} x_{i_1} \ldots x_{i_\beta} x_{j_1} \ldots x_{j_\gamma}) \times \prod_{i=1}^{t} (1 - p^{-s+t+1} x_i) \prod_{i=1}^{t} (1 - x_i),$$

where $\{i_1, \ldots, i_\beta, j_1, \ldots, j_\gamma\}$ runs over all elements of $\Delta(t, \beta, \gamma)$.

Corollary. *For any $1 \leq i \leq t$ the degree of the denominator of $P(T, S; x_1, \ldots, x_s)$ with respect to x_i is $(t-1)2^{t-2} + 2$, and therefore its total degree is $t((t-1)2^{t-2} + 2)$.*

The above theorem can be proved by a careful analysis of the proof of [4, Theorem 1.2] and its corollary. We note that it cannot be derived from the result of [5]. We also note that it gives a more precise result than that of [3]. In fact, put

$$\Gamma(n, \beta, \gamma) = \{(\{i_1, \ldots, i_\beta\}, \{j_1, \ldots, j_\gamma\}) ; 1 \leq i_1 < \ldots < i_\beta \leq n, 1 \leq j_1 < \ldots < j_\gamma \leq n, \{i_1, \ldots, i_\beta\} \cap \{j_1, \ldots, j_\gamma\} = \emptyset\}.$$

We note that $\{i_1, \ldots, i_\beta, j_1, \ldots, j_\gamma\}$ and $(\{i_1, \ldots, i_\beta\}, \{j_1, \ldots, j_\gamma\})$ are distinguished. In fact it happens that $\{i_1, \ldots, i_\beta, j_1, \ldots, j_\gamma\} = \{i'_1, \ldots, i'_\beta, j'_1, \ldots, j'_\gamma\}$ even if $(\{i_1, \ldots, i_\beta\}, \{j_1, \ldots, j_\gamma\}) \neq (\{i'_1, \ldots, i'_\beta\}, \{j'_1, \ldots, j'_\gamma\})$. Then in [2, Theorem 1.1], we have proved that under the same assumption as above the possible denominator of $P(T, S; x_1, \ldots, x_t)$ is

$$(1.1) \qquad \prod_{\beta=1}^{t} \prod_{\gamma=0}^{t-\beta} \prod_{(\{i_1, \ldots, i_\beta\}, \{j_1, \ldots, j_\gamma\})} (1 - p^{\beta(-s+t+\gamma+1)} x_{i_1} \ldots x_{i_\beta} x_{j_1} \ldots x_{j_\gamma}) \prod_{i=1}^{t} (1 - x_i),$$

where $(\{i_1, \ldots, i_\beta\}, \{j_1, \ldots, j_\gamma\})$ runs over all elements of $\Gamma(n, \beta, \gamma)$. Here we make the convention that $x_{i_1} \ldots x_{i_\gamma} = 1$ if $\gamma = 0$. Let $t = 2$, and $s \geq 4$. According to (1.1), the possible denominator of $P(T, S; x_1, x_2)$ is

$$(1 - p^{2(-s+3)} x_1 x_2)(1 - p^{-s+3} x_1)(1 - p^{-s+3} x_2) (1 - p^{-s+4} x_1 x_2)^2 (1 - x_1)(1 - x_2),$$

while by the Theorem 1 in this paper, it is

$$(1 - p^{-s+3} x_1)(1 - p^{-s+3} x_2)(1 - p^{-s+4} x_1 x_2) (1 - x_1)(1 - x_2).$$

Now our next main result is the following:

Theorem 2. *In addition to the notation and the assumption in Theorem 1, assume that S is a unimodular matrix. Then for any $1 \leq i \leq t$ the de-*

gree of $P(T, S; x_1, \ldots, x_t)$ *with respect to* x_i *is at most* -1, *and therefore the total degree of* $P(T, S; x_1, \ldots, x_t)$ *is at most* $-t$.

The above theorem can be also proved by a careful analysis of the proof of [4, Theorem 1.2]. As Corollary to Theorem 1 and Theorem 2, we have

Corollary. *Let the notations be as above. Then the degree of numerator of* $P(T, S; x_1, \ldots, x_t)$ *with respect to* x_i *is at most* $(t-1)2^{t-2}+1$ *for any* $1 \leq i \leq t$, *and therefore its total degree is at most* $t((t-1)2^{t-2}+1)$.

For $T = \mathrm{diag}(b_1, \ldots, b_t)$, we define a formal power series $Q(T, S; x_1, \ldots, x_t)$ by

$$Q(T, S; x_1, \ldots, x_t) = \sum_{r_1, \ldots, r_t = 0}^{\infty} \alpha_p(\mathrm{diag}(p^{r_1}b_1, \ldots, p^{r_t}b_t), S)x_1^{r_1}\ldots x_t^{r_t},$$

which was introduced by Böcherer and Sato [1]. Thus by Theorems 1 and 2, we easily obtain

Theorem 3. (1) *Let the notation and the assumptions be as in Theorem* 1. *Then the possible denominator of* $Q(T, S; x_1, \ldots, x_t)$ *is of the following form*:

$$\prod_{\beta=1}^{t} \prod_{\gamma=1}^{t-\beta} \prod_{\{i_1, \ldots, i_\beta, j_1, \ldots, j_\gamma\}} (1 - p^{\beta(-s+t+\gamma+1)} x_{i_1}^2 \ldots x_{i_\beta}^2 x_{j_1}^2 \ldots x_{j_\gamma}^2)$$
$$\times \prod_{i=1}^{t} (1 - p^{-s+t+1}x_i^2) \prod_{i=1}^{t} (1 - x_i^2),$$

where $\{i_1, \ldots, i_\beta, j_1, \ldots, j_\gamma\}$ *runs over all elements of* $\Delta(t, \beta, \gamma)$.

(2) *Let the notation and the assumptions be as in Theorem* 2. *Then for any* $1 \leq i \leq t$, *the degree of* $Q(T, S; x_1, \ldots, x_t)$ *with respect to* x_i *is at most* -1, *and therefore the total degree of it is at most* $-t$.

We note that (2) of the above theorem gives a certain generalization of the result of [7] on the numerator of the power series.

Example. Let A be a unimodular symmetric matrix of degree $2k$ with entries in $\mathbf{Z}_p$, and b_1 and b_2 p-adic units. Put $\varepsilon = \chi((-1)^k \det A)$, and $\eta = \chi(-b_1b_2)$, where χ is the quadratic character of $\mathbf{Z}_p$ modulo p. Then we have

$$P(B, A; x_1, x_2) = (1-p^{-2k+3}x_1)^{-1}(1-p^{-2k+3}x_2)^{-1}(1-p^{-2k+4}x_1x_2)^{-1}(1-x_1)^{-1}(1-x_2)^{-1}$$
$$\times (1-\varepsilon p^{-k})(1+\varepsilon\eta p^{1-k})R(x_1, x_2),$$

where

$$\begin{aligned} R(x_1, x_2) &= 1 - p^{1-k}\varepsilon\eta(x_1+x_2) + (p^{1-k}\varepsilon\eta + p^{2-k}\varepsilon - p^{3-2k} \\ &\quad - p^{3-2k}\eta + p^{4-3k}\eta\varepsilon + p^{5-3k}\varepsilon)x_1x_2 \\ &\quad - p^{5-3k}\varepsilon x_1x_2(x_1+x_2) + p^{6-4k}\eta x_1^2x_2^2 \\ &= (1-\varepsilon\eta p^{1-k})(1+\varepsilon p^{2-k}x_2)(1-p^{3-2k}x_1x_2) \end{aligned}$$

or

$$\begin{aligned} &= (1-\varepsilon\eta p^{1-k})(1+\varepsilon p^{2-k})(1-p^{3-2k}\eta x_1x_2) \\ &\qquad + p^{3-2k}(\eta+1)(1-x_1)(1-x_2) \end{aligned}$$

according as $B = \mathrm{diag}(b_1, b_2)$, $\mathrm{diag}(b_1, pb_2)$, or $\mathrm{diag}(pb_1, pb_2)$.

The above example may be considered as a reformulation of [6, Theorem 2, (1), (2)]. It shows that Theorems 1 and 2 are best possible.

References

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