A Lusternik-Schnirelmann Type Theorem for Locally Lipschitz Functionals with Applications to Multivalued Periodic Problems

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Abstract: We prove a Lusternik-Schnirelmann type theorem for locally Lipschitz functionals, by replacing the notion of Fréchet-differentiability with the Clarke generalized gradient. We apply our abstract framework to solve a multivalued second order periodic problem generated by non-smooth mappings.

Key words: Locally Lipschitz functional; Clarke subdifferential; Lusternik-Schnirelmann category; multivalued periodic problem.

1. Introduction. In the theory of differential equations two of the most important tools for proving the existence of solutions are the Mountain Pass Theorem of Ambrosetti-Rabinowitz and the Lusternik-Schnirelmann Theorem. These abstract results apply to the case where the solutions of the given problem are critical points of an appropriate functional of energy f, which is supposed to be real and of class C^1, defined on a real Banach space. The case when f fails to be differentiable arises frequently in non-smooth mechanics. In [8] we proved a generalization of the Mountain Pass Theorem for locally Lipschitz functionals. The aim of this paper is to give a variant of the Lusternik-Schnirelmann Theorem for such functionals.

We recall in what follows the main properties of locally Lipschitz functionals. For proofs and further details see [2] or [3].

Throughout, X will be a real Banach space. Let X^* be its dual and $\langle x^*, x\rangle$, for $x \in X$, $x \in X^*$, denote the duality pairing between X^* and X. Let $f : X \to \boldsymbol{R}$ be a locally Lipschitz ($f \in \mathrm{Lip}_{loc}(X, \boldsymbol{R})$). For each x, $v \in X$, we define the generalized directional derivative at x in the direction v of f as

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}$$

The generalized gradient (the Clarke subdifferential) of f at x is the subset $\partial f(x)$ of X^* defined by

$$\partial f(x) = \{x^* \in X^* ; f^0(x, v) \geq \langle x^*, v\rangle, \text{ for all } v \in X\}$$

If f is convex, $\partial f(x)$ coincides with the subdifferential of f at x in the sense of convex analysis.

The fundamental properties of the Clarke subdifferential are:

a) For each $x \in X$, $\partial f(x)$ is a nonempty convex weak-★ compact subset of X^*.

b) For each x, $v \in X$, we have
$f^0(x, v) = \max\{\langle x^*, v\rangle ; x^* \in \partial f(x)\}$

c) The set-valued mapping $x \mapsto \partial f(x)$ is upper semi-continuous in the sense that for each $x_0 \in X$, $\varepsilon > 0$, $v \in X$, there is $\delta > 0$ such that for each $x^* \in \partial f(x)$ with $\| x - x_0 \| < \delta$, there exists $x_0^* \in \partial f(x_0)$ such that $| \langle x^* - x_0^*, v\rangle | < \varepsilon$.

d) The function $f^0(\cdot, \cdot)$ is upper semi-continuous.

e) If f achieves a local minimum or maximum at x, then $0 \in \partial f(x)$.

f) The function

$$\lambda(x) = \min_{x^* \in \partial f(x)} \| x^* \|$$

exists and is lower semi-continuous.

Definition 1. *A point $u \in X$ is said te be a critical point of $f \in \mathrm{Lip}_{loc}(X, \boldsymbol{R})$ if $0 \in \partial f(u)$, namely $f^0(u, v) \geq 0$ for every $v \in X$. A real number c is called a critical value of f if there is a critical point $u \in X$ such that $f(u) = c$.*

2. The main result. Let Z be a discrete subgroup of the real Banach space X, that is

$$\inf_{z \in Z \setminus \{0\}} \| z \| > 0$$

A function $f : X \to \boldsymbol{R}$ is said to be Z-*periodic* if $f(x + z) = f(x)$, for every $x \in X$ and

$z \in Z$.

If $f \in \mathrm{Lip}_{loc}(X, \boldsymbol{R})$ is Z-periodic, then $x \mapsto f^0(x, v)$ is Z-periodic, for all $v \in X$ and ∂f is Z-invariant, that is $\partial f(x+z) = \partial f(x)$, for every $x \in X$ and $z \in Z$. These impliy that λ inherits the Z-periodicity property.

If $\pi : X \to X/Z$ is the canonical surjection and x is a critical point of f, then $\pi^{-1}(\pi(x))$ contains only critical points. Such a set is called a *critical orbit* of f. Note that X/Z is a complete metric space endowed with the metric

$$d(\pi(x), \pi(y)) = \inf_{z \in Z} \| x - y - z \|$$

Definition 2. *A locally Lipschitz Z-periodic function $f : X \to \boldsymbol{R}$ is said to satisfy the $(PS)_Z$-condition provided that, for each sequence (x_n) in X such that $(f(x_n))$ is bounded and $\lambda(x_n) \to 0$, the sequence $(\pi(x_n))$ is relatively compact in X/Z. If c is a real number, then f is said to satisfy the $(PS)_{Z,c}$-condition if, for any sequence (x_n) in X such that $f(x_n) \to c$ and $\lambda(x_n) \to 0$, there is a convergent subsequence of $(\pi(x_n))$.*

We recall somê well-known properties of the Lusternik-Schnirelmann category. See [7] for proofs and details.

Lemma 1. *Let A and B* be *subsets of X. Then the following hold:*

i) *If $A \subset B$, then* $\mathrm{Cat}_X(A) \le \mathrm{Cat}_X(B)$

ii) $\mathrm{Cat}_X(A \cup B) \le \mathrm{Cat}_X(A) + \mathrm{Cat}_X(B)$

iii) *Let $h : [0,1] \times A \to X$ be a continuous mapping such that $h(0,x) = x$ for every $x \in A$. If A is closed and $B = h(1, A)$, then* $\mathrm{Cat}_X(A) \le \mathrm{Cat}_X(B)$

iv) *If n is the dimension of the vector space generated by the discrete group Z, then , for each $1 \le i \le n+1$, the set*

$\mathscr{A}_i = \{A \subset X ; A$ is compact and $\mathrm{Cat}_{\pi(X)}\pi(A) \ge i\}$ *is nonempty. Obviously, $\mathscr{A}_1 \supset \mathscr{A}_2 \supset \dots \supset \mathscr{A}_{n+1}$.*

The following two Lemmas are proved in [9].

Lemma 2. *For each $1 \le j \le n+1$, the space $\mathscr{A}_i$ endowed with the Hausdorff metric*

$$\delta(A, B) = \max\{\sup_{a \in A} \mathrm{dist}(a, B), \sup_{b \in B} \mathrm{dist}(b, A)\}$$

is a complete metric space.

Lemma 3. *If $1 \le i \le n+1$ and $f \in C(X, \boldsymbol{R})$, then the function $\eta : \mathscr{A}_i \to \boldsymbol{R}$ defined by*

$$\eta(A) = \max_{x \in A} f(x)$$

is lower semi-continuous.

Let $f : X \to \boldsymbol{R}$ be a Z-periodic locally Lipschitz function with the $(\mathrm{RS})_Z$-property. Moreover, we suppose that f is bounded below. We shall denote by $\mathrm{Cr}(f, c)$ the set of critical points of f at the level $c \in \boldsymbol{R}$, that is

$$\mathrm{Cr}(f, c) = \{x \in X ; f(x) = c \text{ and } \lambda(x) = 0\}$$

For each $c \in \boldsymbol{R}$ we denote $[f \le c] = \{x \in X ; f(x) \le c\}$.

Theorem 1. *Let $f : X \to \boldsymbol{R}$ be a bounded below Z-periodic locally Lipschitz function which satisfies the $(PS)_Z$-condition.*

Then f has at least $n+1$ distinct critical orbits, where n is the dimension of the vector space generated by the discrete group Z.

Proof. For each $1 \le i \le n+1$, let

$$c_i = \inf_{A \in \mathscr{A}_i} \eta(A)$$

It follows from Lemma 1 *iv*) and the lower boundedness of f that

$$-\infty < c_1 \le c_2 \le \dots \le c_{n+1} < +\infty$$

It is sufficient to show that, if $1 \le i \le j \le n+1$ and $c_i = c_j = c$, then the set $\mathrm{Cr}(f, c)$ contains at least $j - i + 1$ distinct critical orbits. We argue by contradiction and suppose that, for some $i \le j$, $\mathrm{Cr}(f, c)$ has $k \le j - i$ distinct critical orbits, generated by $x_1, \dots, x_k \in X$. We construct first an open neighbourhood of $\mathrm{Cr}(f, c)$ of the form

$$V_r = \bigcup_{l=1}^{k} \bigcup_{z \in Z} B(x_l + z, r)$$

Moreover, we may suppose that $r > 0$ is chosen such that π is one-to-one on $\bar{B}(x_l, 2r)$. This condition ensures that $\mathrm{Cat}_{\pi(X)}(\pi(\bar{B}(x_l, 2r))) = 1$, for each $l = 1, \dots, k$. Here $V_r = \emptyset$ if $k = 0$.

Step 1. We prove that there exists $0 < \varepsilon < \min\left\{\frac{1}{4}, r\right\}$ such that, for each $x \in [c - \varepsilon \le f \le c + \varepsilon] \setminus V_r$, one has

$$\lambda(x) > \sqrt{\varepsilon} \tag{1}$$

Indeed, if not, there is a sequence (x_m) in $X \setminus V_r$ such that, for each $m \ge 1$,

$$c - \frac{1}{m} \le f(x_m) \le c + \frac{1}{m} \text{ and } \lambda(x_m) \le \frac{1}{\sqrt{m}}$$

Since f satisfies $(PS)_Z$, it follows that, up to a subsequence, $\pi(x_m) \to \pi(x)$ as $m \to \infty$, for some $x \in X \setminus V_r$. By the Z-periodicity of f and λ, we can assume that $x_m \to x$ as $m \to \infty$. The continuity of f and the lower semi-continuity of λ imply $f(x) = c$ and $\lambda(x) = 0$, which is a contradiction, since $x \in X \setminus V_r$.

Step 2. For ε found above and according to

the definition of c_j, there exists $A \in \mathscr{A}_j$ such that
$$\max_{x \in A} f(x) < c + \varepsilon^2$$
Setting $B = A \setminus V_{2r}$, we get by Lemma 1 that
$$j \leq \mathrm{Cat}_{\pi(X)}(\pi(A)) \leq \mathrm{Cat}_{\pi(X)}(\pi(B) \cup \pi(\bar{V}_{2r})) \leq$$
$$\leq \mathrm{Cat}_{\pi(X)}(\pi(B)) + \mathrm{Cat}_{\pi(X)}(\pi(\bar{V}_{2r})) \leq$$
$$\mathrm{Cat}_{\pi(X)}(\pi(B)) + k \leq \mathrm{Cat}_{\pi(X)}(\pi(B)) + j - i$$
Hence, $\mathrm{Cat}_{\pi(X)}(\pi(B)) \geq i$, that is $B \in \mathscr{A}_i$.

Step 3. For ε and B as above we apply the Ekeland's Principle to the functional η defined in Lemma 3. It follows that there exists $C \in \mathscr{A}_i$ such that, for each $D \in \mathscr{A}_i$, $D \neq C$,
$$\eta(C) \leq \eta(B) \leq \eta(A) \leq c + \varepsilon^2$$
$$\delta(B, C) \leq \varepsilon$$
$$\eta(D) > \eta(C) - \varepsilon\delta(C, D) \tag{2}$$
Since $B \cap V_{2r} = \emptyset$ and $\delta(B, C) \leq \varepsilon < r$, it follows that $C \cap V_r = \emptyset$. In particular, the set $F = [c - \varepsilon \leq f] \cap C$ is contained in $[c - \varepsilon \leq f \leq c + \varepsilon]$ and $F \cap V_r = \emptyset$. Applying Lemma 1 in [8] to $\varphi = \partial f$ on F, we find a continuous map $v : F \to X$ such that, for all $x \in F$ and $x^* \in \partial f(x)$,
$$\| v(x) \| \leq 1 \text{ and } \langle x^*, v(x)\rangle \geq \inf_{x \in F} \lambda(x) - \varepsilon \geq$$
$$\inf_{x \in C} \lambda(x) - \varepsilon \geq \sqrt{\varepsilon} - \varepsilon$$
where the last inequality is justified by (1).

It follows that, for each $x \in F$ and $x^* \in \partial f(x)$,
$$f^0(x, -v(x)) = \max_{x^* \in \partial f(x)} \langle x^*, -v(x)\rangle =$$
$$-\max_{x^* \in \partial f(x)} \langle x^*, v(x)\rangle \leq \varepsilon - \sqrt{\varepsilon} < -\varepsilon,$$
from our choice of ε.

From the upper semi-continuity of f^0 and the compactness of F, there exists $\delta > 0$ such that if $x \in F$, $y \in X$, $\| y - x \| \leq \delta$, then
$$f^0(y, -v(x)) < -\varepsilon \tag{3}$$
Since $C \cap \mathrm{Cr}(f, c) = \emptyset$ and C is compact, while $\mathrm{Cr}(f, c)$ is closed, there is a continuous extension $w : X \to X$ of v such that $w|_{\mathrm{Cr}(f,c)} = 0$ and $\| w(x) \| \leq 1$, for all $x \in X$.

Let $\alpha : X \to [0,1]$ be a continuous Z-periodic function such that $\alpha = 1$ on $[f \geq c]$ and $\alpha = 0$ on $[f \geq c - \varepsilon]$. Let $h : [0,1] \times X \to X$ be the continuous mapping defined by
$$h(t, x) = x - t\delta\alpha(x)w(x)$$
If $D = h(1, C)$, it follows from Lemma 1 that
$$\mathrm{Cat}_{\pi(X)}(\pi(D)) \geq \mathrm{Cat}_{\pi(X)}(\pi(C)) \geq i$$
which shows that $D \in \mathscr{A}_i$, since D is compact.

Step 4. By Lebourg's mean value theorem (see [4]) we get that, for each $x \in X$, there exists $\theta \in (0,1)$ such that
$$f(h(1, x)) - f(h(0, x)) \in \langle \partial f(h(\theta, x)), -\delta\alpha(x)w(x)\rangle$$
Hence, there is some $x^* \in \partial f(h(\theta, x))$ such that
$$f(h(1, x)) - f(h(0, x)) = \alpha(x)\langle x^*, -\delta w(x)\rangle$$
It follows from (3) that, if $x \in F$, then
$$\begin{aligned} f(h(1, x)) - f(h(0, x)) &= \delta\alpha(x)\langle x^*, -w(x)\rangle \\ &\leq \delta\alpha(x) f^0(x - \theta\delta\alpha(x)w(x), -v(x)\rangle \\ &\leq -\varepsilon\delta\alpha(x) \end{aligned} \tag{4}$$
It follows that, for each $x \in C$,
$$f(h(1, x)) \leq f(x)$$
Let $x_0 \in C$ be such that $f(h(1, x_0)) = \eta(D)$. Hence,
$$c \leq f(h(1, x_0)) \leq f(x_0)$$
By the definition of α and F, it follows that $\alpha(x_0) = 1$ and $x_0 \in F$. Therefore, by (4), we get
$$f(h(1, x_0)) - f(x_0) \leq -\varepsilon\delta$$
Thus,
$$\eta(D) + \varepsilon\delta \leq f(x_0) \leq \eta(C) \tag{5}$$
Taking into account the definition of D, it follows that
$$\delta(C, D) \leq \delta$$
Therefore,
$$\eta(D) + \varepsilon\delta(C, D) \leq \eta(C)$$
so that (2) implies $C = D$, which contradicts (5). □

3. An application. We shall study the periodic multivalued problem of the forced-pendulum
$$\begin{cases} x''(t) + f(t) \in [\underline{g}(x(t)), \bar{g}(x(t))] & \text{a.e. } t \in (0,1) \\ x(0) = x(1) \end{cases} \tag{6}$$
where
$$f \in L^p(0,1) \quad \text{for some } p > 1 \tag{7}$$
$$g \in L^\infty(\mathbf{R}),\ g(x + T) = g(x) \tag{8}$$
for some $T > 0$, a.e. $x \in \mathbf{R}$
$$\underline{g}(s) = \lim_{\varepsilon \searrow 0} \operatorname{essinf} \{g(t)\,;\, |t - s| < \varepsilon\} \tag{9}$$
$$\bar{g}(s) = \lim_{\varepsilon \searrow 0} \operatorname{esssup} \{g(t)\,;\, |t - s| < \varepsilon\}$$
$$\int_0^T g(t)\,dt = \int_0^T f(t)\,dt = 0 \tag{10}$$

Theorem 2. *If f, g are as above, then the problem* (6) *has at least two solutions in* $X := H_p^1(0,1) = \{x \in H^1(0,1)\,;\, x(0) = x(1)\}$, *which are distinct in the sense that their difference is not an integer multiple of T.*

Sketch of the proof. The critical points of the locally Lipschitz map

$\varphi : X \to \boldsymbol{R} \quad \varphi(x) = -\frac{1}{2}\int_0^1 x'^2 + \int_0^1 fx - \int_0^1 G(x)$

are solutions of (6), where $G(t) = \int_0^t g(s)ds$.

Since $\varphi(x+T) = \varphi(x)$, we are going to use Theorem 1. We shall verify only that φ has the $(\mathrm{PS})_{Z,c}$-property, for each real c. The details of the proof and further results will appear elsewhere.

Let $(x_n) \subset X$ be such that

(11) $\varphi(x_n) \to c$

(12) $\lambda(x_n) \to 0$

Let $w_n \in \partial\varphi(x_n) \subset L^\infty(0,1)$ (since $\underline{g} \circ x_n \le w_n \le \bar{g} \circ x_n$ and $\underline{g}, \bar{g} \in L^\infty(\boldsymbol{R})$) be such that

$$\lambda(x_n) = x_n'' + f - w_n \to 0 \text{ in } H^{-1}(0,1)$$

Then, multiplying (12) by x_n we get

$$\int_0^1 (x_n')^2 - \int_0^1 fx_n + \int_0^1 w_n x_n = o(1)\, \| x_n \|_{H_p^1}$$

and, by (11),

$$-\frac{1}{2}\int_0^1 (x_n')^2 + \int_0^1 fx_n - \int_0^1 G(x_n) \to c,$$

so that there exist positive constants C_1, C_2 such that

$$\int_0^1 (x_n')^2 \le C_1 + C_2 \| x_n \|_{H_p^1}$$

Note that G is also T-periodic, hence bounded.

Replacing x_n by $x_n + kT$ for a suitable integer k, we may suppose that

$$x_n(0) \in [0, T]$$

so that (x_n) is bounded in H_p^1.

Let $x \in H_p^1$ be such that, up to a subsequence, $x_n \to x$ and $x_n(0) \to x(0)$.
Then

$$\int_0^1 (x_n')^2 = \langle -x_n'' - f + w_n, x_n - x\rangle + \int_0^1 w_n(x_n - x) - \int_0^1 f(x_n - x) + \int_0^1 x_n' x' \to \int_0^1 x'^2$$

because $x_n \to x$ in $L^{p'}$, where p' is the conjugated exponent of p.

It follows that $x_n \to x$ in H_p^1. □

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