Wavelet Transforms Associated to a Principal Series Representation of Semisimple Lie Groups. I

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1. Introduction. Let G be a locally compact Lie group and π a continuous representation of G on a Hilbert space $\mathscr{H}$. Let $\mathscr{H}_\infty$ denote the space of C^∞-vectors in $\mathscr{H}$, endowed with a natural Sobolev-type topology, and $\mathscr{H}_{-\infty}$ the dual of $\mathscr{H}_\infty$ endowed with the strong topology. We denote the corresponding representation on $\mathscr{H}_{-\infty}$ by the same letter π. Let S be a subset of G and ds a measure on S. A vector $\psi \in \mathscr{H}_{-\infty}$ is said to be S-strongly admissible for π if there exists a positive constant $c_{S,\psi}$ such that

$$(1) \qquad \int_S |\langle f, \pi(s)\psi\rangle_{\mathscr{H}}|^2 ds = c_{S,\psi} \|f\|^2_{\mathscr{H}} \quad \text{for all } f \in \mathscr{H}_\infty,$$

where $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{H}}$ denote the inner product and the norm of $\mathscr{H}$ respectively. We easily see that $\psi \in \mathscr{H}_{-\infty}$ is S-strongly admissible for π if and only if, as a functional on $\mathscr{H}_\infty$,

$$(2) \qquad f = c_{S,\psi}^{-1} \int_S \langle f, \pi(s)\psi\rangle_{\mathscr{H}} \pi(s)\psi ds \quad \text{for all } f \in \mathscr{H}_\infty.$$

We call $\langle f, \pi(s)\psi\rangle$ the wavelet transform of f associated to (G, π, S, ψ) in the sense that, by specializing (G, π, S, ψ), the above formula yields a group theoretical interpretation of various well-known wavelet transforms. For example, we first let $S = G$, $ds = dg$, a Haar measure of G, and $(\pi, \mathscr{H})$ a square-integrable representation of G, that is, π is an irreducible unitary representation satisfying $0 < \int_G |\langle\phi, \pi(g)\psi\rangle|^2 dg < \infty$ for all ϕ, ψ in $\mathscr{H}$. Then π is a discrete series of G and every $\psi \in \mathscr{H}$ is a G-strongly admissible vector for π (see [3]). The Gabor transform and the Grossmann-Morlet transform correspond to the Weyl-Heisenberg group and the one-dimensional affine group respectively (cf. [7, §3]). Next let H be a closed subgroup of G and π a discrete series of G/H. Then there exists an H-invariant distribution vector $\psi \in \mathscr{H}_{-\infty}$ for which (2) holds by replacing S and ds with G/H and a G-invariant measure on G/H respectively (cf. [12]). We can treat this case in our scheme, because the integral over G/H can be regarded as the one over $S = \sigma_0(G/H)$ where $\sigma_0 : G/H \to G$ is a flat section of the fiber bundle $G \to G/H$.

These considerations are based on the existence of the discrete series of G or G/H, so it seems to be difficult to unfold the same process in the case that G has no such representations. One approach to treat the case is to find a non flat Borel section $\sigma : G/H \to G$. In the case of the Poincaré group and the affine Weyl-Heisenberg group, Ali, Antoine, and Gazeau [1] and Kalisa and Torrésani [10] respectively find a non square-integrable reprepentation $(\pi, \mathscr{H})$, a ψ in $\mathscr{H}$, and a non flat section σ such that (2) holds for π, ψ, and $S = \sigma(G/H)$. In this paper we shall investigate a transform associated to a principal series representation of noncompact semisimple Lie groups and we obtain a generalization of the Grossmann-Morlet transform and the Carderón identity. A transform associated to the analytic continuation of the holomorphic discrete series and its limit will be treated in the forthcoming paper [9].

2. Principal series representations. Let G be a noncompact connected semisimple Lie group with finite center and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_0 + \mathfrak{n}_0$ an Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G. According to the process in [4, §6], we shall define a standard parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$. Let Σ be the set of roots of $(\mathfrak{g}, \mathfrak{a}_0)$ positive for $\mathfrak{n}_0$ and Σ_0 the subset of Σ consisting of simple roots. For each $F \subset \Sigma_0$ we set $\mathfrak{a} = \mathfrak{a}_F = \{H \in \mathfrak{a}_0 ; \alpha(H) = 0 \text{ for all } \alpha \in F\}$ and $\mathfrak{n} = \mathfrak{n}_F = \sum_{\alpha\in\Sigma\setminus\Sigma_F} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the root space corresponding to α. Then the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is given by $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ where $Z_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{m}$

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$+\mathfrak{a}$. The set of roots of $(\mathfrak{g}, \mathfrak{a})$ positive for $\mathfrak{n}$ is given by $\Sigma(\mathfrak{a}) = \{\alpha^{\sim}; \alpha \in \Sigma\}$ where $\alpha^{\sim} = \alpha|_{\mathfrak{a}}$, and let $\rho = \sum_{\alpha \in \Sigma(\mathfrak{a})} \alpha/2$. We denote by K, A_0, M_0, A, and N the analytic subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{a}_0$, $\mathfrak{m}$, $\mathfrak{a}$, and $\mathfrak{n}$ respectively. The parabolic subgroup P of G corresponding to $\mathfrak{p}$ is given by $P = MAN$ where $M = Z_K(\mathfrak{a})M_0$. We denote by θ the Cartan involution of G and put $\bar{N} = \theta(N)$. Haar measures dg, dm, dn, and $d\bar{n}$ of G, M, N, and $\bar{N}$ are respectively normalized as the following integral formula holds: for $f \in L^1(G)$

(3) $$\int_G f(g)dg = \iiiint_{\bar{N}\times M\times A\times N} f(\bar{n}man) e^{2\rho(\log a)} d\bar{n}dmdadn,$$

where da is the Lebesgue measure on A (see [5, §19]). For $\lambda \in \mathfrak{a}_C^*$, the dual space of the complexification of $\mathfrak{a}$, we define $1 \otimes e^{\lambda} \otimes 1(man) = (man)^{\lambda} = e^{\lambda(\log a)}$ and let $\pi_\lambda = \mathrm{Ind}_P^G(1 \otimes e^{\lambda} \otimes 1)$. A dense subspace of the representation space $H(\lambda)$ is

(4) $$\{f \in C(G)\,;\, f(gman) = e^{-(i\lambda+\rho)(\log a)} f(g) \quad (g \in G,\ man \in MAN)\}$$

with norm $\|f\|^2 = \int_K |f(k)|^2 dk$. By restricting f to $\bar{N}$, we see that $H(\lambda)$ is identified with $L^2(\bar{N}, e^{-2\Im\lambda(H(\bar{n}))}d\bar{n})$ and the action of G is given by

(5) $$\pi_\lambda(g)f(\bar{n}) = e^{(i\lambda+\rho)\log a(g^{-1}\bar{n})} f(\bar{n}(g^{-1}\bar{n})),$$

where $g = kma^{H(g)}n \in G = KMAN$ and $g = \bar{n}(g)ma(g)n \in \bar{N}MAN$. Then π_λ is unitary if and only if $\lambda \in \mathfrak{a}^*$ (see [6, §4]). Let $\mathcal{S}(\bar{N})$ be the Schwartz space on $\bar{N}$ and $\mathcal{S}'(\bar{N})$ the dual space with respect to

$$\langle f, g\rangle_{L^2(\bar{N})} = \int_K f(k)\bar{g}(k)dk.$$

3. Plancherel formula for $L^2(\bar{N})$. General theory of the Plancherel formula on nilpotent Lie groups (cf. [2, 4.3.10]) yields that

(6) $$\|\phi\|^2_{L^2(\bar{N})} = \int_{U\cap V_T} \|\sigma_\omega(\phi)\|^2_{HS}\,\mu(\omega)d\omega \quad \text{for all } \phi \in L^2(\bar{N}).$$

Here U is the set of generic coadjoint orbits, V_T a subspace of $\bar{\mathfrak{n}}^*$, σ_ω the irreducible unitary representation of $\bar{N}$ corresponding to $\omega \in U$, and $\sigma_\omega(\phi)$ the operator defined by $\sigma_\omega(\phi) = \int_{\bar{N}} \phi(\bar{n})\sigma_\omega(\bar{n})d\bar{n}$. Moreover, $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm and $\mu(\omega)d\omega$ the Plancherel measure on $U \cap V_T$. Since U is Zariski open, we may replace $U \cap V_T$ in (6) by V_T or V_T', the set of regular elements in V_T. We here note that

(7) $$\sigma_\omega(\mathrm{Ad}(s)\bar{n}) \sim \sigma_{\mathrm{Ad}(s^{-1})\omega}(\bar{n}) \quad (s \in MA),$$

where $\mathrm{Ad}(s^{-1})\omega(\bar{n}) = \omega(\mathrm{Ad}(s)\bar{n})$ (cf. [2, 2.1.3]) and $\sigma_\omega(\psi)$ is well-defined for $\psi \in \mathcal{S}'(\bar{N})$ as an operator on $\mathcal{S}(\bar{N})$.

4. Main theorem. Let S be a measurable subset of MA and ds a measure on S. We suppose that there exists $\psi \in \mathcal{S}'(\bar{N})$ satisfying for all $\omega \in V_T'$

(*i*) $\sigma_\omega(\psi)\sigma_\omega(\psi)^* = n_\psi(\omega)I$,

(*ii*) $0 < \int_S n_\psi(\mathrm{Ad}(s^{-1})\omega)ds = c_{S,\psi} < \infty$

where $n_\psi(\omega)$ is a real number, I is the identity operator, and $c_{S,\psi}$ is independent of ω.

Theorem 1. *Let $\psi \in \mathcal{S}'(\bar{N})$ be as above and suppose $\lambda|_S = -i\rho|_S$. Then ψ is a $\bar{N}S$-strongly admissible vector for π_λ, that is,*

$$\iint_{\bar{N}\times S} |\langle f, \pi_\lambda(\bar{n}s)\psi\rangle_{L^2(\bar{N})}|^2 d\bar{n}ds = c_{S,\psi}\|f\|^2_{L^2(\bar{N})}$$

for all $f \in \mathcal{S}(\bar{N})$.

Proof. We first recall that, since $\lambda|_S = -i\rho|_S$,

$$\pi_\lambda(s)\psi(\bar{n}) = \psi(\mathrm{Ad}(s^{-1})\bar{n}s^{-1}) = \psi(\mathrm{Ad}(s^{-1})\bar{n})s^{2\rho}.$$

Then, it follows from (*i*) and (7) that

(8) $$n_{\pi_\lambda(s)\psi}(\omega) = n_\psi(\mathrm{Ad}(s^{-1})\omega).$$

Therefore, (*i*), (*ii*), (6), and (8) yields that for $f \in \mathcal{S}(\bar{N})$

$$\iint_{\bar{N}\times S} |\langle f, \pi_\lambda(\bar{n}s)\psi\rangle_{L^2(\bar{N})}|^2 d\bar{n}ds$$
$$= \iint_{\bar{N}\times S} |f * (\pi_\lambda(s)\psi)^{\sim}(\bar{n})|^2\, d\bar{n}ds \qquad (\psi^{\sim}(\bar{n}) = \bar{\psi}(\bar{n}^{-1}))$$
$$= \int_S\int_{V_T'} \|\sigma_\omega(f * (\pi_\lambda(s)\psi)^{\sim})\|^2_{HS}\mu(\omega)d\omega ds$$
$$= \int_{V_T'} \mathrm{Tr}\Big(\sigma_\omega(f)\cdot\int_S \sigma_\omega(\pi_\lambda(s)\psi)^*\sigma_\omega(\pi_\lambda(s)\psi)ds\cdot \sigma_\omega(f)^*\Big)\mu(\omega)d\omega$$
$$= c_{S,\psi}\int_S \|\sigma_\omega(f)\|^2_{HS}\,\mu(\omega)d\omega$$
$$= c_{S,\psi}\|f\|^2_{L^2(\bar{N})}. \qquad \square$$

Similarly, we can deduce the following,

Theorem 2. *Let $\psi \in \mathcal{S}'(\bar{N})$ be as above and suppose $\lambda|_S \equiv 0$. Then, ψ is a $S\bar{N}$-strongly admissible vector for π_λ, that is,*

$$\iint_{S\times\bar{N}} |\langle f, \pi_\lambda(s\bar{n})\psi\rangle_{L^2(\bar{N})}|^2 dsd\bar{n} = c_{S,\psi}\|f\|^2_{L^2(\bar{N})}$$

for all $f \in \mathcal{S}(\bar{N})$.

Remark 3. The conclution in Theorem 1 is equivalent to the following identity:

$$f = c_{S,\phi}^{-1} \int_S f * (\pi_\lambda(s)\phi)^{\sim} * \pi_\lambda(s)\phi ds$$
for all $f \in \mathcal{S}(\bar{N})$.

We may regard this identity as a generalization of the Carderón identity (cf. [11, p.16]).

5. Examples. We recall a basis realization of $\sigma_\omega (\omega \in V_T')$ (cf. [2, 4.1.1]). Let $\mathfrak{m}$ be a polarizing subalgebra for *all* ω (we abuse $\mathfrak{m}$ in $\mathfrak{p}$) and $\{X_1, \ldots, X_m, \ldots, X_n\}$ a weak Malcev basis for $\bar{\mathfrak{n}}$ passing through $\mathfrak{m}$ where $n = \dim \bar{\mathfrak{n}}$ and $m = \dim \mathfrak{m}$. If we put $k = m - n$ and define $\gamma(t) = \exp t_1 X_{m+1} \ldots \exp t_k X_n$ for $t = (t_1, \ldots, t_k) \in \boldsymbol{R}^k$, then $\gamma : \boldsymbol{R}^k \to G$ is a cross-section for $M \backslash G$ ($M = \exp \mathfrak{m}$), and the Lebesgue measure dt on $\boldsymbol{R}^k$ corresponds to a G-invariant measure on $M \backslash G$. Then, σ_ω is realized on $L^2(\boldsymbol{R}^k)$ as $\sigma_\omega(\bar{n}) f(t) = e^{2\pi i \omega(X(\gamma(t)\bar{n}))} f(t(\gamma(t)\bar{n}))$ where $\bar{n} = \exp X(\bar{n}) \gamma(t(\bar{n}))$ ($X(\bar{n}) \in \mathfrak{m}$, $t(\bar{n}) \in \boldsymbol{R}^k$), and $\sigma_\omega(\phi)$ ($\phi \in \mathcal{S}'(\bar{N})$) is the operator with the kernel given by $K_\phi(t', t) = \int_M \chi_\omega(m) \phi(\gamma(t')^{-1} m \gamma(t)) dm$ where $\chi_\omega(\exp Y) = e^{2\pi i \omega(Y)}$ for $Y \in \mathfrak{m}$ (cf. [2, 4.2.2]). We here assume that

(A1) $\mathfrak{m}$ is ideal and $\bar{\mathfrak{n}}/\mathfrak{m}$ is abelian.

Then, $K_\phi(t', t) = \int_{\mathfrak{m}} e^{2\pi i \omega(\mathrm{Ad}(\gamma(t'))Y)} \phi(\exp Y \gamma(t - t')) dY$. We now specialize $\phi \in \mathcal{S}'(\bar{N})$ by letting $\phi(\bar{n}) = \Psi(X(\bar{n})) \Xi(t(\bar{n}))$ where $\Psi \in \mathcal{S}'(\boldsymbol{R}^m)$ and $\Xi \in \mathcal{S}'(\boldsymbol{R}^k)$ satisfy

(9) $|\hat{\Psi}(\mathrm{Ad}(\gamma(t))\omega)| = |\hat{\Psi}(\omega)|$ and $|\hat{\Xi}(t)| = 1$ for all $t \in \boldsymbol{R}^k$

respectively. Since $K_\phi(t', t) = \hat{\Psi}(\mathrm{Ad}(\gamma(-t'))\omega) \Xi(t - t')$, $\sigma_\omega(\phi)$ satisfies $\sigma_\omega(\phi) f(t') = \hat{\Psi}(\mathrm{Ad}(\gamma(-t'))\omega) \Xi^{\sim} * f(t')$ and hence, $n_\phi(\omega) = |\hat{\Psi}(\omega)|^2$ in (*i*). Next we identify V_T with $\boldsymbol{R}^r$ by using coroots vectors. Then we assume that there exists a subgroup A_1 of A_0 such that $\dim A_1 = r$ and

(A2) $da = \dfrac{dx}{|x|}$

for $x = \mathrm{Ad}(a)\omega$ ($a \in A_1$, $\omega \in V_T$),

where $|x| = \Pi_{i=1}^r |x_i|$. Let $\mathscr{E}$ denote the set of signatures $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ where $\varepsilon_i = \pm 1$ for $1 \le i \le r$, and D_ε the domain in $\boldsymbol{R}^r$ defined by $D_\varepsilon = \{x \in \boldsymbol{R}^r ; 0 < \varepsilon_i x_i < \infty (1 \le i \le r)\}$. Since $n_\phi(\omega) = |\hat{\Psi}(\omega)|^2$ and $\int_{A_1} n_\phi(\mathrm{Ad}(a)\omega) da = \int_{D\mathrm{sgn}\omega} n_\phi(x) dx/|x|$ for $\omega \in V_T'$, the condition (*ii*) for $S = A_1$ can be rewritten as

(10) $0 < \int_{D_\varepsilon} |\hat{\Psi}(x)|^2 \dfrac{dx}{|x|} = c_\Psi < \infty$ for all $\varepsilon \in \mathscr{E}$,

where c_Ψ is independent of ε. Therefore, under (A1) and (A2) the conditions (*i*) and (*ii*) hold for $\phi = \Psi\Xi$ satisfying (10). For example, when (a) $G = SL(n+2, \boldsymbol{R})$ ($n \ge 1$), $\bar{N} = H_n$, the $(2n+1)$-dimensional Heisenberg group, and $A_1 = \{\mathrm{diag}(a, 1, \ldots, 1, a^{-1}) ; a \in \boldsymbol{R}_+\}$, and (b) $G = SL(4, \boldsymbol{R})$, $\bar{N} = N_4$, the group of lower triangular 4×4 matrices with 1's along the diagonal, and $A_1 = \{\mathrm{diag}(a, b, b^{-1}, a^{-1}) ; a, b \in \boldsymbol{R}_+\}$, we can show (A1) and (A2) and moreover, we can find Ψ and Ξ satisfying (9) and (10) (see [8]).

Remark 4. For a nonempty subset $\mathscr{L}$ of $\mathscr{E}$, we define $\mathcal{S}^{\mathscr{L}}(\bar{N}) = \{f \in \mathcal{S}(\bar{N}) ; \sigma_\omega(f) \equiv 0$ if $\mathrm{sgn}\omega \notin \mathscr{L}\}$ and instead of (10) we suppose that Ψ in (9) satisfies

(11) $$\int_{D_\varepsilon} |\hat{\Psi}(x)|^2 \frac{dx}{|x|} = \begin{cases} c_{\mathscr{L},\Psi} & \text{if } \varepsilon \in \mathscr{L} \\ 0 & \text{otherwise,} \end{cases}$$

where $c_{\mathscr{L},\Psi}$ is nonzero finite and independent of $\varepsilon \in \mathscr{L}$. Then Theorem 1 and Theorem 2 respectively hold for $\mathcal{S}^{\mathscr{L}}(\bar{N})$.

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