A Uniform Construction of the Root Lattices E_6, E_7, E_8 and their Dual Lattices*)

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A simple construction of the root lattice E_r and its dual lattice E_r^* is given, which works uniformly with respect to the rank $r = 6,7,8$. An advantage of this construction is that the description of the minimal vectors in E_r and E_r^* is reasonably concise; for instance, the 240 roots in E_8 can be enumerated in a few lines. (Compare with standard references such as [1], [2, Ch.4, §8], [3, Ch. 4]).

Our construction is inspired by 1) the classical theory of del Pezzo surfaces, according to which a del Pezzo surface of degree $d = 1,2,3$ is a blowing up of $r = 9 - d$ points from the projective plane $\mathbf{P}^2$ (cf. [3]), and 2) the viewpoint of Mordell-Weil lattices (cf. [4], [5], [6], [7]).

1. Construction.

Definition. Let L_r be a free $\mathbf{Z}$-module of rank $r = 6,7,8$ generated by r elements $u_1, \ldots, u_r$, and define a symmetric bilinear pairing on L_r by the rule:

$$\langle u_i, u_j \rangle = \delta_{ij} + \frac{1}{d} \tag{1}$$

where we set $d = 9 - r$ so that $d = 3,2,1$ according as $r = 6,7,8$.

Proposition 1. *L_r is a positive-definite lattice of rank r such that*

$$\det L_r = 1 + \frac{r}{d} = \begin{cases} 3 & (r = 6) \\ \frac{9}{2} & (r = 7) \\ 9 & (r = 8) \end{cases} \tag{2}$$

Proof. This is an immediate consequence of the following:

Lemma 2. *Suppose that $A = (a_{ij})$ is a real symmetric matrix of degree n such that $a_{ij} = \delta_{ij} + s$ for all i, j for a fixed positive number s. Then $\det A = 1 + ns$. In particular, such a matrix A is always positive-definite.*

Proof. Note that each line sums up to $1 + ns$. Hence

$$\det A = \begin{vmatrix} 1+s & s & \cdots & s \\ s & 1+s & \cdots & s \\ \vdots & \vdots & \ddots & \vdots \\ s & s & \cdots & 1+s \end{vmatrix} = \begin{vmatrix} 1+s & s & \cdots & s \\ s & 1+s & \cdots & s \\ \vdots & \vdots & \ddots & \vdots \\ 1+ns & 1+ns & \cdots & 1+ns \end{vmatrix}$$

$$= (1+ns) \begin{vmatrix} 1+s & s & \cdots & s \\ s & 1+s & \cdots & s \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} = (1+ns) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} = 1 + ns.$$

This proves the first assertion. For any $i = 1,2, \ldots, n$, the i-th principal minor of A has $\det = 1 + is > 0$. Hence A is a positive-definite matrix. Q.E.D.

Definition. Let $\tilde{L}_r$ be the $\mathbf{Z}$-submodule of $L_r \otimes \mathbf{Q}$ generated by L_r and the following element v_0:

$$v_0 = \frac{1}{3} \sum_{i=1}^{r} u_i. \tag{3}$$

Proposition 3. *By naturally extending the pairing $\langle , \rangle$ to $\tilde{L}_r$, it becomes a positive-definite lattice of rank r such that $[\tilde{L}_r : L_r] = 3$. We have*

$$\det \tilde{L}_r = \frac{1}{3^2} \det L_r = \begin{cases} \frac{1}{3} & (r = 6) \\ \frac{1}{2} & (r = 7) \\ 1 & (r = 8) \end{cases} \tag{4}$$

Proof. In general, if U is a sublattice of index ν in a lattice V, then we have $\det V = \det U / \nu^2$. Hence the result. Q.E.D.

Note that

*) To the memory of Akira Okada.

$$(5)\quad \langle v_0, u_j\rangle = \frac{1}{3}\sum_i \langle u_i, u_j\rangle$$

$$= \frac{1}{3}\left(1+\frac{r}{d}\right) = \begin{cases} 1 & (r=6) \\ \frac{3}{2} & (r=7) \\ 3 & (r=8) \end{cases}$$

and

$$(6)\quad \langle v_0, v_0\rangle = \frac{r}{3}\cdot\frac{1}{3}\left(1+\frac{r}{d}\right) = \begin{cases} 2 & (r=6) \\ \frac{7}{2} & (r=7) \\ 8 & (r=8) \end{cases}$$

Definition. We set

$$(7)\qquad \alpha_{ij} = u_i - u_j \quad (i \neq j)$$

$$(8)\quad \beta_{ijk} = v_0 - (u_i + u_j + u_k) \quad (i, j, k \text{ distinct})$$

Let L_r^0 denote the sublattice of $\tilde{L}_r$ generated by the r elements:

$$(9)\quad \alpha_i = \alpha_{i\,i+1}\ (i = 1, \dots, r-1),\ \beta = \beta_{123}.$$

Lemma 4. (i) *Each element α_{ij} or β_{ijk} has norm* 2:

$$(10)\qquad \langle \alpha_{ij}, \alpha_{ij}\rangle = 2,\ \langle \beta_{ijk}, \beta_{ijk}\rangle = 2.$$

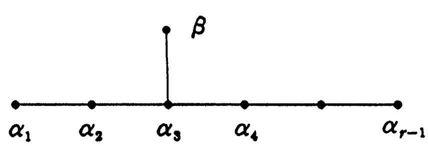
(ii) *For the r elements in* (9), *we have* for $i \neq j$

$$(11)\qquad \langle \alpha_i, \alpha_j\rangle = \begin{cases} -1 & (|i-j| = 1) \\ 0 & \text{otherwise} \end{cases},$$

$$\langle \beta, \alpha_i\rangle = \begin{cases} -1 & (i = 3) \\ 0 & \text{otherwise} \end{cases}$$

Proof. This is easily checked by using (1), (5), (6). Q.E.D.

The above lemma shows that $\{\alpha_i, \beta\}$ forms a basis of the root system of type E_r, associated with the Dynkin diagram (cf. [1], [3]).



Theorem 5. *For any $r = 6,7,8$, the lattice L_r^0 is isomorphic to the root lattice E_r, and $\tilde{L}_r$ is isomorphic to the dual lattice E_r^* of E_r.*

Proof. Since L_r^0 is generated by $\{\alpha_i, \beta\}$ which forms a basis of the root system of type E_r, it is clearly isomorphic to the root lattice E_r. In particular, we have $\det L_r^0 = \det E_r = d = 3,2$ or 1 according as $r = 6,7$ or 8. This implies that the index of L_r^0 in $\tilde{L}_r$ is d.

On the other hand, we claim that $\tilde{L}_r$ is contained in the dual lattice $(L_r^0)^*$ of L_r^0. It suffices to check this for generators $\{u_i, v_0\}$ of $\tilde{L}_r$. By (1)-(8), we have

$$(12)\qquad \langle u_i, \alpha_j\rangle = \delta_{ij} - \delta_{i\,j+1} \in \mathbf{Z}$$

$$(13)\quad \langle u_i, \beta\rangle = -(\delta_{i1} + \delta_{i2} + \delta_{i3}) \in \mathbf{Z}.$$

Hence each u_i is contained in $(L_r^0)^*$. Similarly, we have

$$(14)\qquad \langle v_0, \alpha_j\rangle = 0,\quad \langle v_0, \beta\rangle = -1.$$

This shows $v_0 \in (L_r^0)^*$.

Thus we have proved that $\tilde{L}_r$ is contained in $(L_r^0)^*$. By noting that the index $[L_r^0 : \tilde{L}_r] = d$ is equal to $[E_r : E_r^*]$, we conclude that $\tilde{L}_r = (L_r^0)^* \simeq E_r^*$. Q.E.D.

Corollary 6. *The orthogonal complement $\langle v_0\rangle^\perp$ of v_0 in $L_r \otimes \mathbf{Q}$ is generated by $\{\alpha_i\}$ in* (9).

Proof. This is immediate from (14). Q.E.D.

Remark. In case $r = 8$, we see directly that $\tilde{L}_8$ is an even integral lattice. Indeed, its generators satisfy $\langle u_i, u_i\rangle = 2$ and $\langle v_0, v_0\rangle = 8$. Hence $\tilde{L}_8$ is a positive-definite even unimodular lattice of rank 8. As is wellknown, such a lattice is unique up to isomorphism, and this gives another proof of the fact $\tilde{L}_8 = L_8^0 \simeq E_8$.

2. Minimal vectors. From now on, we make the identification:

$$L_r^0 = E_r,\quad L_r = E_r^*.$$

We keep the same notation as before: u_i, v_0, α_{ij}, $\beta_{ijk}, \dots$

Now we describe the minimal vectors of E_r^* in terms of u_i, v_0. Also we determine the positive roots of E_r with respect to the chosen basis $\{\alpha_i, \beta\}$.

Let us introduce some more elements of E_r^*. We set

$$(15)\qquad u_i' = u_i - v_0 \ (i = 1, \dots, r),$$

$$(16)\qquad \gamma_{ij} = v_0 - u_i - u_j \ (i \neq j).$$

The norm of these vectors can be easily computed. We have

$$(17)\qquad \langle u_i', u_i'\rangle = \begin{cases} \frac{4}{3} & (r=6) \\ 2 & (r=7) \\ 4 & (r=8) \end{cases}$$

$$(18)\qquad \langle \gamma_{ij}, \gamma_{ij}\rangle = \begin{cases} \frac{4}{3} & (r=6) \\ \frac{3}{2} & (r=7) \\ 2 & (r=8) \end{cases}$$

Case $r = 6$. For $r = 6$, it is known that there are 72 roots (of norm 2) in E_6 and that there are 54 minimal vectors of minimal norm $4/3$ in E_6^*.

Theorem 7. *For $r = 6$, consider the following set:*

(19) $\Omega = \{u_i, u_i' = u_i - v_0 \ (i = 1, \ldots, 6),$
$\gamma_{ij} = v_0 - u_i - u_j \ (i < j)\}.$

Then (i) it consists of $6 + 6 + 15 = 27$ minimal vectors of E_6^*, and the union of Ω and $-\Omega$ gives all the 54 minimal vectors of E_6^*.

(ii) The Weyl group $W(E_6)$ acts transitively on the set Ω, and Ω and $-\Omega$ are the two orbits of $W(E_6)$ in the set of minimal vectors.

(iii) The 72 roots of E_6 are given by $\pm v_0$, $\alpha_{ij} (i \neq j)$, and $\pm \beta_{ijk} (i < j < k)$.

(iv) *The following elements*

(20) $-v_0, \alpha_{ij} (i < j), \beta_{ijk} (i < j < k)$

give $1 + 15 + 20 = 36$ positive roots with respect to the basis $\{\alpha_i, \beta\}$ in (9).

Proof. By (1), (17), (18), u_i, u_i', γ_{ij} have norm $4/3$, i.e. they are minimal vectors in E_6^*. Next all the elements in Ω and $-\Omega$ are distinct; this is immediate if we look at their expression as $\mathbf{Q}$-linear combination of $u_1, \ldots, u_r$. Hence (i) follows.

Similarly (iii) follows from (6) and (10).

To show (iv), we check that each element in (20) is a linear combination of α_i, β with non-negative (integer) coefficients. In fact, we have

(21) $-v_0 = 2\beta + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5.$

(22) $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \ (i < j)$

(23) $\beta_{ijk} = \beta + \cdots \ (i < j < k).$

Here $\cdots$ stands for a linear combination of α_l. Of the above relations, (21) and (22) are obvious. For (23), note that $\beta_{ijk} - \beta$ is orthogonal to v_0, and hence it is a linear combination of α_l by Corollary 6. Further the coefficients must be necessarily nonnegative, since the coefficient of β is $1 > 0$. (Recall that any root in a root lattice is either positive or negative, i.e. the coefficients in terms of a basis are simultaneously nonnegative or nonpositive.)

For (ii), recall ([1]) that the Weyl group $W(E_r) \subset \mathrm{Aut}(E_r)$ is the subgroup generated by the reflections s_α which are defined as follows: for each root α, we let

$$s_\alpha(x) = x - \langle x, \alpha \rangle \alpha \ (x \in E_r).$$

To prove (ii), we show in the lemma below that $s_\alpha(\Omega) \subset \Omega$ for $\alpha = \alpha_m (m < r = 6)$ or β in the basis. Since such s_α generate $W(E_r)$, Ω is acted on by the Weyl group. Further, this action is transitive, as is clear from the lemma.

Lemma 8. (i) *Suppose $\alpha = \alpha_m (m < 6)$. Then s_α permutes the elements in each set $\{u_i\}$, $\{u_i'\}$, $\{\gamma_{ij}\}$ among themselves. More explicitly, we have*

(24) $$s_\alpha(u_i) = \begin{cases} u_i \\ u_{i+1}, \\ u_{i-1} \end{cases}$$

$$s_\alpha(u'_i) = \begin{cases} u_i' & (i \neq m, m+1) \\ u_{i+1}' & (i = m) \\ u_{i-1}' & (i = m+1). \end{cases}$$

Let $M = \{m, m+1\}$. Then, for $\gamma_{ij} (i < j)$, we have

(25) $$s_\alpha(\gamma_{ij}) = \begin{cases} \gamma_{ij} & \text{if } \{ij\} = M \text{ or } \{ij\} \cap M = \emptyset \\ \gamma_{i\pm1\, j} & \text{if } i \in M, j \bar{\in} M \\ \gamma_{i\, j\pm1} & \text{if } i \bar{\in} M, j \in M. \end{cases}$$

(ii) *Next suppose $\alpha = \beta = \beta_{123}$, and set $J = \{123\}$, $K = \{456\}$. Then*

(26) $$s_\beta(u_i) = \begin{cases} u_i & (i \in K) \\ \gamma_{jk} & (i \in J = \{ijk\}), \end{cases}$$

$$s_\beta(u_i') = \begin{cases} u_i' & (i \in J) \\ \gamma_{jk} & (i \in K = \{ijk\}) \end{cases}$$

(27) $$s_\beta(\gamma_{ij}) = \begin{cases} u_k & (i, j \in J = \{ijk\}) \\ u_k' & (i, j \in K = \{ijk\}) \\ \gamma_{ij} & \text{otherwise} \end{cases}$$

Proof. The verification is straightforward and is omitted.

This completes the proof of Theorem 7.

Remark. Theorem 7 is closely related to the theory of the 27 lines on a cubic surface. The set Ω can be put in a bijective correspondence with the set of 27 lines, and $\{u_i, u_i' \ (i = 1, \ldots, 6)\}$ plays the role of *double six* in Schläfli's sense. For this, we refer to [7]; cf. [3].

Case r = 7. It is known that there are 126 roots (of norm 2) in the root lattice E_7 and 56 minimal vectors of minimal norm $3/2$ in the dual lattice E_7^*.

Theorem 9. (i) *The minimal vectors of E_7^* are given by*

(28) $\pm u_i \ (i = 1, \ldots, 7),$
$\pm \gamma_{ij} = \pm(v_0 - u_i - u_j) \ (i < j).$

(ii) *The positive roots of E_7 are given by the following $7 + 21 + 35 = 63$ elements:*

(29) $u_i' = u_i - v_0, \alpha_{ij} (i < j), \beta_{ijk} (i < j < k).$

Proof. First (i) follows from (1) and (17). In the same way as the proof of Th. 7 (iv), we have

(30) $u_i' = 2\beta + \cdots$

(31) $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \ (i < j)$

(32) $\beta_{ijk} = \beta + \cdots (i < j < k)$
where $\cdots$ indicates a linear combination of α_i with nonnegative integral coefficients. Q.E.D.

Remark. For the connection to algebraic geometry, we refer to [3] and [6].

Case r = 8. It is known that there are 240 roots (of norm 2) in E_8, which are at the same time the minimal vectors of $E_8^* = E_8$.

Theorem 10. *The following* $8 + 28 + 56 + 28 = 120$ *elements*:

(33) $\quad -u_i (i = 1, \ldots, 8),\ \alpha_{ij}(i < j),$
$\quad \beta_{ijk}(i < j < k),\ -\gamma_{ij}(i < j)$

are the positive roots of E_8 *with respect to the basis* $\{\beta, \alpha_i (i < 8)\}$.

Proof. All these elements have norm 2 by (1), (10) and (18), and they (and their minus) are distinct elements in E_8. Further, by the same arguments as the proof of Th. 7(iv), we have

(34) $-u_i = 3\beta + \cdots$

(35) $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}\ (i < j)$

(36) $\beta_{ijk} = \beta + \cdots\ (i < j < k)$

(37) $-\gamma_{ij} = 2\beta + \cdots\ (i < j)$

with the unwritten part $\cdots$ meaning as usual some linear combination of α_i with nonnegative integral coeffients. Q.E.D.

Remark. In [1], a root of the form $b\beta + \sum_i c_i\alpha_i \in E_r$ is denoted by the symbol

$$\begin{matrix} c_1 & c_2 & c_3 & c_4 & \cdots & c_{r-1} \\ & & b & & & \end{matrix},$$

which will be denoted here by $(b\,; c_1\ c_2\ c_3 \cdots c_{r-1})$ for the sake of easier printing. Then the above (34) can be written out as follows:

$-u_1 = (3; 1354321),\ -u_2 = (3; 2354321)$
$-u_3 = (3; 2454321),\ -u_4 = (3; 2464321)$
$-u_5 = (3; 2465321),\ -u_6 = (3; 2465421)$
$-u_7 = (3; 2465431),\ -u_8 = (3; 2465432).$

These roots are listed at the last part in the table of positive roots of E_8 in [1], Planche VII (II), presumably because they have the most complicated coefficients. The corresponding facts hold for the cases $r = 6,7$.

As it turns out, our uniform construction for $r = 6,7,8$ may be said to go the other way around, since we start from $u_1, \ldots, u_r$.

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