Relations of the Weyl Groups of Extended Affine Root Systems $A_l^{(1,1)}$, $B_l^{(1,1)}$, $C_l^{(1,1)}$, $D_l^{(1,1)}$

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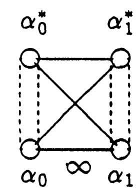
1. Introduction. Extended affine root systems were introduced and classified by K. Saito, related to simple elliptic singularity, and extended affine Weyl groups have been also studied in [1], [2] from the view point of algebraic geometry. Further to study extended affine Weyl groups from the view point of representation theory, as a first step in the previous paper ([5]) we examined the defining relations of extended affine Weyl groups. Further in this paper using the result, according to the suggestion by K. Saito, we describe the relations of extended affine Weyl groups from the view point of a generalization of a Coxeter system. In the cases of finite on affine root sys'tems, which are called a Coxeter system, and well known ([3], [4]), but in the cases of the extended affine root systems, they are not a Coxeter system and the set of roots corresponding to the Dynkin diagram are not linearly independent. However by considering all reflections corresponding to Dynkin diagram, we can describe them by using the figure of a diagram, and the relation are very natural and beautiful.

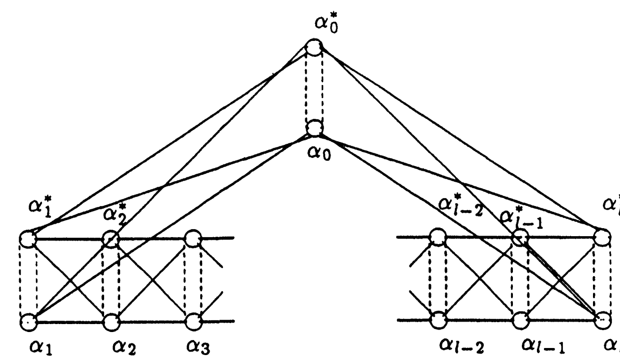
2. Relations of extended affine Weyl group. Let F be a real vector space of finite rank with a symmetric bilinear form $I : F \times F \to R$. A subset Φ of F is caalled an extended affine root system, if I is positive semi definite and the radical $rad(I) = \{x \in F \mid I(x, y) = 0 \text{ for } \forall_y \in F\}$ is of rank 2 over R and satisfy a system of a axiom for a generalized root system belong to I ([1]). Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{l+1}$ are orthonormal vectors in a real vector space F, and $rad(I) = Ra \oplus Rb$. Then the root system Φ and Dynkin diagram of $A_l^{(1,1)}$, $B_l^{(1,1)}$, $C_l^{(1,1)}$, $D_l^{(1,1)}$ are given as follows ([1]);

$A_l^{(1,1)}$, $\Phi = \{\pm (\varepsilon_i - \varepsilon_j) + nb + ma \quad (1 \leq i < j \leq l+1), (n, m \in Z)\}$,

$A_1^{(1,1)}$

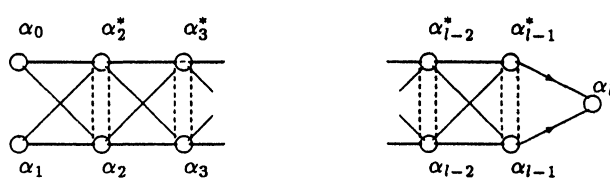


$A_l^{(1,1)}$, $(l \geq 2)$



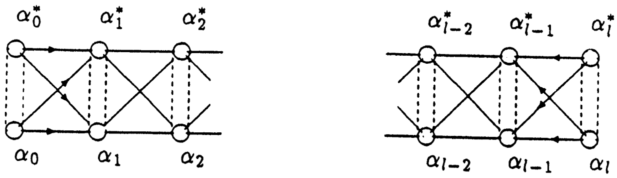
$B_l^{(1,1)}$ $(l \geq 3)$,

$\Phi = \{\pm \varepsilon_i + nb + ma \ (1 \leq i \leq l) \ (n, m \in Z)$,
$\pm \varepsilon_i \pm \varepsilon_j + nb + ma \ (1 \leq i < j \leq l) \ (n, m \in Z)\}$.

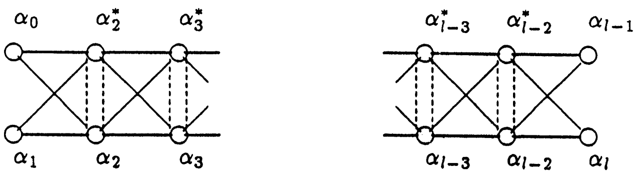


$C_l^{(1,1)}$ $(l \geq 2)$,

$\Phi = \{\pm 2\varepsilon_i + nb + ma \ (1 \leq i \leq l) \ (n, m \in Z)$,
$\pm \varepsilon_i \pm \varepsilon_j + nb + ma \ (1 \leq i < j \leq l), (n, m \in Z)\}$.



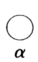
$D_l^{(1,1)}$ $(l \geq 4)$,
$\Phi = \{\pm \varepsilon_i \pm \varepsilon_j + nb + ma \ (1 \leq i < j \leq l), (n, m \in Z)\}$.



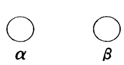
In the above diagrams, let $w_i := w_{\alpha_i}$ and $w_i^* := w_{\alpha_i^*}$ be the reflection corresponding to each root α_i and α_i^* respectively, then we have;

Theorem 2.1. *The relations of* w_i *and* w_i^* *are given as follows: for any* $\alpha, \beta, \gamma \in \{\alpha_0, \alpha_1, \cdots, \alpha_l, \alpha_0^*, \alpha_1^*, \cdots, \alpha_l^*\}$

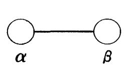
(*i*)



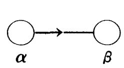
$\Rightarrow \ w_\alpha^2 = 1$



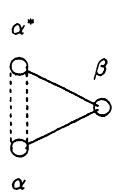
$\Rightarrow \ (w_\alpha w_\beta)^2 = 1$



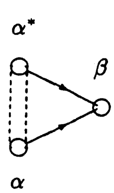
$\Rightarrow \ (w_\alpha w_\beta)^3 = 1$



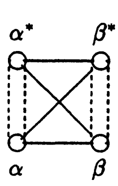
$\Rightarrow \ (w_\alpha w_\beta)^4 = 1$



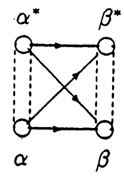
$\Rightarrow (w_\alpha w_\beta w_\alpha^* w_\beta)^3 = 1$



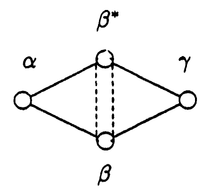
$\Rightarrow (w_\alpha w_\beta w_\alpha^* w_\beta)^2 = 1$



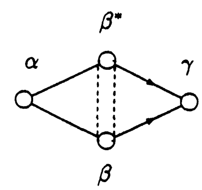
$\Rightarrow$ $w_\alpha w_\beta^* w_\alpha = w_\beta w_\alpha^* w_\beta$, *and* $w_\alpha^* w_\beta w_\alpha^* = w_\beta^* w_\alpha w_\beta^*$



$\Rightarrow \ w_\alpha^* w_\beta w_\alpha^* = w_\alpha w_\beta^* w_\alpha$



$\Rightarrow \ (w_\alpha w_\beta^* w_\alpha w_\beta w_\gamma w_\beta)^2 = 1$



$\Rightarrow \ (w_\alpha w_\beta^* w_\alpha w_\beta w_\gamma w_\beta)^2 = 1$

(*ii*)

$A_l^{(1,1)} \Rightarrow w_0 w_0^* w_1 w_1^* \cdots\cdots w_{l-1} w_{l-1}^* w_l w_l^* = 1$

$B_l^{(1,1)} \Rightarrow (w_0 w_1 w_2 w_2^* w_3 w_3^* \cdots\cdots w_{l-1} w_{l-1}^* w_l)^2 = 1$

$C_l^{(1,1)} \Rightarrow w_0 w_0^* w_1 w_1^* \cdots\cdots w_{l-1} w_{l-1}^* w_l w_l^* = 1$

$D_l^{(1,1)} \Rightarrow (w_0 w_1 w_2 w_2^* w_3 w_3^* \cdots\cdots w_{l-2} w_{l-2}^* w_{l-1} w_l)^2 = 1.$

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