Integrability of Infinitesimal Automorphisms of Linear Poisson Manifolds

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1. Introduction. In the present paper, we discuss the integrability of infinitesimal automorphisms of linear Poisson manifolds. An infinitesimal automorphism X is said to be *integrable,* if it is a *Hamiltonian vector field.*

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. The linear Poisson structure on $\mathfrak{g}^*$ is defined as a Lie algebra structure on $C^\infty(\mathfrak{g}^*)$ satisfying Leibniz identity. This is equal to giving an antisymmetric contravariant 2-tensor P on $\mathfrak{g}^*$ which satisfies Jacobi identity. More precisely, for all $f, g \in C^\infty(\mathfrak{g}^*)$ and $\mu \in \mathfrak{g}^*$, the Poisson bracket is given by

$$\{f, g\}(\mu) = \langle \mu, [d_\mu f, d_\mu g] \rangle,$$

where $[\ ,\ ]$ is the Lie algebra operation in $\mathfrak{g}$, $\langle , \rangle$ is the pairing of $\mathfrak{g}^*$ with $\mathfrak{g}$, and $d_\mu f$ is the differential of f considered as an element of $\mathfrak{g}$ instead of $\mathfrak{g}^{**}$. In the case of general Poisson manifolds, the Poisson bracket is given by $\{f, g\} = \langle P \mid df \wedge dg \rangle$.

We denote by $G \cdot \mu$ the G-orbit passing through $\mu \in \mathfrak{g}^*$ with respect to the coadjoint representation of G on $\mathfrak{g}^*$. By the theorem of Kirillov-Kostant-Souriau, each $G \cdot \mu$ is a symplectic leaf in $\mathfrak{g}^*$. (Hence it is even dimensional.) Let G_μ be the isotropy group at μ. Then $G \cdot \mu$ is diffeomorphic to G/G_μ. For more informations about linear Poisson manifolds, see [7].

Now we shall define three (infinite dimensional) Lie algebras of vector fields on $\mathfrak{g}^*$. By an infinitesimal automorphism of $\mathfrak{g}^*$, we mean a smooth vector field X on $\mathfrak{g}^*$ such that $L(X)P = 0$, where $L(X)$ denotes the Lie derivative along X. We denote by $\mathcal{L}$ the Lie algebra consisting of such vector fields X. Let $\mathcal{I}$ be a Lie subalgebra of $\mathcal{L}$ consisting of vector fields X such that each X is tangent to symplectic leaves $G \cdot \mu$. Given $f \in C^\infty(\mathfrak{g}^*)$, $\{f, \cdot\}$ defines a derivation of $C^\infty(\mathfrak{g}^*)$. Hence there corresponds a vector field ξ_f, which we call the *Hamiltonian vector field.* And we denote by $\mathcal{H}$ the Lie algebra of Hamiltonian vector fields. Then there are canonical inclusions: $\mathcal{L} \supset \mathcal{I} \supset \mathcal{H}$. Direct calculation shows that both Lie subalgebras $\mathcal{I}$ and $\mathcal{H}$ are ideals of $\mathcal{L}$.

A vector field X of $\mathcal{L}$ is called "integrable" if it belongs to $\mathcal{H}$. If all vector fields of $\mathcal{L}$ are integrable (i.e. $\mathcal{L} = \mathcal{H}$), then $\mathcal{L}$ is called integrable. In the case of $\mathfrak{g} = \mathfrak{so}(3, R)$, we proved that $\mathcal{L}$ is integrable ([3] and [4]). In this paper, we treat the case of $\mathfrak{g} = \mathfrak{sl}(2, R)$.

Recall that the quotient space $\mathcal{L}/\mathcal{H}$ is nothing but the first Poisson cohomology ([1] and [5]). There are many papers about Poisson cohomology of "regular" Poisson manifolds ([1], [5], [6] and [8]). Note that linear Poisson manifolds give typical examples of "nonregular" Poisson manifolds. Therefore our study can be regarded as the first approach to the study of Poisson cohomology of "nonregular" Poisson manifolds.

2. Chevalley-Eilenberg complex. In this section, we shall express the integrability of vector fields in terms of Lie algebra cohomology (see for example [4]). Let (V, ρ) be any representation of the Lie algebra $\mathfrak{g}$ on a vector space. Associated to this representation, there is the Chevalley-Eilenberg complex:

$$V \xrightarrow{\partial_0} V \otimes \Lambda^1 \mathfrak{g}^* \xrightarrow{\partial_1} V \otimes \Lambda^2 \mathfrak{g}^*$$

where coboundary operators are defined by setting

$$(\partial_0 \alpha)(\xi_1) = \rho(\xi_1)(\alpha),$$
$$(\partial_1 \beta)(\xi_1 \wedge \xi_2) = \rho(\xi_1)(\beta(\xi_2)) - \rho(\xi_2)(\beta(\xi_1)) - \beta([\xi_1, \xi_2]),$$

for all $\alpha \in V$ and $\beta \in V \otimes \Lambda^1 \mathfrak{g}^*$ and $\xi_1, \xi_2 \in \mathfrak{g}$. It holds that $\partial_1 \cdot \partial_0 = 0$. The quotient $H^1(\mathfrak{g}; (V, \rho)) = \text{kernel } (\partial_1)/\text{image } (\partial_0)$ is called the first cohomology group of $\mathfrak{g}$ with coefficients in the module (V, ρ). Recall that when $\mathfrak{g}$ is semisimple and V is finite dimensional, the space $H^1(\mathfrak{g}; (V, \rho))$ vanishes. Let $x_1, x_2, \ldots, x_n$ be the basis of $\mathfrak{g}$. Then $x_1, x_2, \ldots, x_n$ are considered as coordinate functions on $\mathfrak{g}^*$. We denote by $F\{\mathfrak{g}\}$ the space of all formal functions with variables $x_1, x_2, \ldots, x_n$. $F\{\mathfrak{g}\}$ can be identified with the set of all formal Taylor expansions of functions $f \in C^\infty(\mathfrak{g}^*)$ at the origin. Put $V = F\{\mathfrak{g}\}$. Then $F\{\mathfrak{g}\} \otimes \Lambda^1\mathfrak{g}^*$ is the Lie algebra of all formal vector fields on $\mathfrak{g}^*$, and $F\{\mathfrak{g}\}$ is a projective limit of finite dimensional $\mathfrak{g}$-modules. Hence the first cohomology group still vanishes when $\mathfrak{g}$ is semisimple.

We shall define the action of $\mathfrak{g}$ on the space $F\{\mathfrak{g}\}$ by

$$\rho(x_i)(f) = \sum_{1 \leqslant j,k \leqslant n} C_{ij}^k x_k \frac{\partial f}{\partial x_j},$$

for any $f \in F\{\mathfrak{g}\}$. Then ρ is the representation of $\mathfrak{g}$ on $F\{\mathfrak{g}\}$. A direct computation shows that the map $\eta \to L(\eta)P$ is identified with the coboundary operator in degree 1:

$$\partial_1 : F\{\mathfrak{g}\} \otimes \Lambda^1\mathfrak{g}^* \to F\{\mathfrak{g}\} \otimes \Lambda^2\mathfrak{g}^*.$$

Similarly, the map $g \mapsto \xi_g$ is identified with the coboundary operator in degree 0:

$$\partial_0 : F\{\mathfrak{g}\} \to F\{\mathfrak{g}\} \otimes \Lambda^1 g^*.$$

Kernel (∂_1) is the space of all formal infinitesimal automorphisms and image (∂_0) is the space of all formal Hamiltonian vector fields. Since $H^1(\mathfrak{g}, (F\{\mathfrak{g}\}, \rho)) = 0$ when $\mathfrak{g}$ is semisimple, the integrability problem was affirmatively solved in the formal category. Namely, every formal infinitesimal automorphism is necessarily a formal Hamiltonian vector field.

Proposition A ([4]). *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then for any element X of $\mathscr{L}$, there exists an element ξ_f of $\mathscr{H}$ such that the formal expansion of X at the origin coincides with that of ξ_f.*

A C^∞-vector field X is said to be *flat* (at the origin) if the formal Taylor expansion of X at the origin vanishes. The above proposition states that for any element X of $\mathscr{L}$, there exists an element ξ_f of $\mathscr{H}$ such that $X - \xi_f$ is flat at the origin, if $\mathfrak{g}$ is semisimple.

3. Results. Throughout this section, $\mathfrak{g} = \mathfrak{sl}(2, R)$. We can choose $x, y, z \in \mathfrak{g}$ as coordinates functions on $\mathfrak{sl}(2, R)^*$ as follows:

$$\{x, y\} = -z, \ \{y, z\} = x, \ \{z, x\} = y.$$

Hence the linear Poisson structure P on $\mathfrak{sl}(2, R)^*$ is given by

$$P = -z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}.$$

With respect to the coadjoint action of the Lie group $SL(2, R)$ on $\mathfrak{sl}(2, R)^*$, coadjoint orbits are origin, hyperboloid of one sheet, of two sheet, and circular conics. Every Casimir function is of the form $\phi(t)$, $t = x^2 + y^2 - z^2$, where $\phi(t)$ is a C^∞-function of one variable. The space of Casimir functions is denoted by $\mathscr{C}$, and the subspace of Casimir functions which are flat at the origin is denoted by $\mathscr{C}'$.

Identifying $\mathfrak{sl}(2, R)^*$ with R^3, the pair (R^3, P) can be considered as a linear Poisson manifold. We denote by L the set of formal vector fields which leave the Poisson structure P invariant. Let $F\{\mathfrak{g}\}$ be the space of formal functions on R^3. Then $L \subset F\{\mathfrak{g}\} \otimes \mathfrak{g}^*$. Here the basis of $\mathfrak{g}^*$ is $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$. Put $H = \{X_f \mid f \in F\{\mathfrak{g}\}\}$. By the remark stated in the previous section, we have $L = H$. E. Borel showed that formal Taylor expansion mapping $C^\infty(\mathfrak{g}^*) \to F\{\mathfrak{g}\}$ is surjective. Hence $\mathscr{H} \to H$ is surjective and moreover $\mathscr{L} \to H = L$ is also surjective. From now on, we denote this linear mapping by $\tilde{T}$ and put $\ker \tilde{T} = \mathscr{L}'$. ($\mathscr{L}'$ is an ideal of $\mathscr{L}$.) Under these notations, we prove some lemmas.

Lemma 1 ([2]). *Let $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$ be a smooth vector field on R^3.*

(i) *X belongs to $\mathscr{L}$ if and only if there exists $\phi(t) \in \mathscr{C}$ such that $xf + yg - zh = \phi(x^2 + y^2 - y^2)$, and $\mathrm{div}(X) = 2\phi'(x^2 + y^2 - z^2)$, where $\mathrm{div}(X) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$.*

(ii) *X belongs to $\mathscr{I}$ if and only if $xf + yg - zh = 0$ and $\mathrm{div}(X) = 0$.*

Lemma 2. *Let $\alpha(t)$ and $\beta(t)$ be C^∞-functions which are flat at the origin. Put*

$$m(t) = \begin{cases} 0, & t \leqq 0, \\ \alpha(t), & t > 0. \end{cases} \quad s(t) = \begin{cases} \beta(t), & t \leqq 0, \\ 0, & t > 0. \end{cases}$$

Then $m(t)$ and $s(t)$ are C^∞-functions, and the following two vector fields X_1 and X_2 belong to $\mathscr{L}'$:

$$X_1 = \frac{x \cdot m(x^2 + y^2 - z^2)}{x^2 + y^2}\frac{\partial}{\partial x} + \frac{y \cdot m(x^2 + y^2 - z^2)}{x^2 + y^2}\frac{\partial}{\partial y},$$

$$X_2 = \frac{y \cdot s(x^2 + y^2 - z^2)}{y^2 - z^2}\frac{\partial}{\partial y} + \frac{z \cdot s(x^2 + y^2 - z^2)}{y^2 - z^2}\frac{\partial}{\partial z}.$$

Proof. Use Lemma 1 and then direct calculations. Q.E.D.

Lemma 3. (i) $\mathscr{L} = \mathscr{H} + \mathscr{L}'$. (ii) $\mathscr{L}/\mathscr{H} \cong \mathscr{L}'/\mathscr{L}' \cap \mathscr{H}$.

Proof. (i) For any element X of $\mathscr{L}$, there exists $Y \in \mathscr{H}$ such that $T(X) = T(Y)$. Hence $X - Y \in \mathscr{L}'$. (ii) is clear from (i). Q.E.D.

By Lemma 1, we get a linear mapping $A : X \in \mathscr{L} \to \phi \in \mathscr{C}$. For this linear mapping A, we have

Lemma 4. *$A(\mathscr{L}') \subset \mathscr{C}'$ and $A|_{\mathscr{L}'} : \mathscr{L}' \to \mathscr{C}'$ is surjective.*

Proof. Let $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$ be an element of $\mathscr{L}'$. Then f, g, h are flat at the origin. Put $F(x, y, z) = \phi(x^2 + y^2 - z^2) = xf + yg - zh$. Since for any positive integer k, $\frac{\partial^{2k} F}{\partial x^{2k}}(0,0,0) = 0$, it follows that $\phi^{(k)}(0) = 0$. Hence we have $\phi(t) \in \mathscr{C}'$. Conversely, for $\phi(t) \in \mathscr{C}'$, put

$$m(t) = \begin{cases} 0, & t \leqq 0, \\ \phi(t), & t > 0. \end{cases} \quad s(t) = \begin{cases} \phi(t), & t \leqq 0, \\ 0, & t > 0. \end{cases}$$

Then $m(t)$ and $s(t)$ are C^∞-functions. And if we choose X_1 and X_2 as in Lemma 2, then $X = X_1 + X_2 \in \mathscr{L}'$ and we get $A(X) = \phi$. Q.E.D.

Note that $\ker(A|_{\mathscr{L}'}) = \mathscr{L}' \cap \mathscr{I}$. Then we have $\mathscr{L}'/(\mathscr{L}' \cap \mathscr{I}) = \mathscr{C}'$ by Lemma 4. On the other hand, it holds $(\mathscr{L}'/\mathscr{L}' \cap \mathscr{H})/(\mathscr{L}' \cap \mathscr{I}/\mathscr{L}' \cap \mathscr{H}) \cong \mathscr{L}'/(\mathscr{L}' \cap \mathscr{I}) \cong \mathscr{C}'$. Thus we have only to determine the structure of the space $(\mathscr{L}' \cap \mathscr{I})/(\mathscr{L}' \cap \mathscr{H})$.

Let $B : \mathscr{L}' \cap \mathscr{I} \to H^1(\mathscr{F}, R)$ be a linear mapping defined by $B(X) = [i(X)\omega]$, where $\mathscr{F}$ is an arbitrary hyperboloid of one sheet and ω is a symplectic form on $\mathscr{F}$. For $X \in \mathscr{L}' \cap \mathscr{I}$, since $i(X)\omega$ is a closed 1-form on $\mathscr{F}$, the mapping B is well-defined. We should note that it does not depend on the choice of $\mathscr{F}$, whether $[i(X)\omega]$ becomes a generator of $H^1(\mathscr{F}, R)$.

Proposition B. $\ker(B) = \mathscr{L}' \cap \mathscr{H}$.

Outline of Proof. It is clear that $\ker(B) \supset \mathscr{L}' \cap \mathscr{H}$. We prove the converse. Let $X \in \ker(B)$. If $P = (x, y, z)$ is a point on a hyperboloid of one sheet $\mathscr{F}_1 : x^2 + y^2 - z^2 = c^2$ $(c > 0)$, put

$$F(x, y, z) = -\int_{\gamma_1} i(X)\omega,$$

where γ_1 is a path on $\mathscr{F}_1$ joining (c, 0,0) with $P = (x, y, z)$. Since $X \in \ker(B)$, the value of the right hand side does not depend on the choice of γ_1. If $P = (x, y, z)$ is a point on a hyperboloid of two sheets $\mathscr{F}_2 : x^2 + y^2 - z^2 = -c^2$, $(c > 0)$, put

$$F(x, y, z) = -\int_{\gamma_2} i(X)\omega.$$

Here if $z > 0$, γ_2 is a path on the upper hyperboloid of two sheets $\mathscr{F}_2$ joining $(c, 0,0)$ with $P = (x, y, z)$ and, if $z < 0$, it is a path on the lower hyperboloid of two sheets $\mathscr{F}_2$ joining $(-c, 0,0)$ with $P = (x, y, z)$. Since each sheet is simply connected, the above integral does not depend on the choice of γ_2. On the subset $R^3 -$ {circular conics}, it clearly holds that the function $F(x, y, z)$ is smooth and $X = X_F$.

Let $Q = (x, y, z)$ be a point on the circular conics. (Q may be the origin.) Then the value $\lim_{P_1 \to Q} F(P_1)$ is completely decided, where P_1 is a point on $\mathscr{F}_1$. Similarly, the value $\lim_{P_2 \to Q} F(P_2)$ is also completely decided, where P_2 is on $\mathscr{F}_2$. Moreover, we get $\lim_{P_1 \to Q} F(P_1) = \lim_{P_2 \to Q} F(P_2)$. Hence we can define the new function $\tilde{F}(x, y, z)$ by

$$\tilde{F}(x, y, z) = \begin{cases} F(x, y, z), \text{ if } (x, y, z) \in R^3 \\ \qquad - \{\text{circular conics}\}, \\ \lim_{P_1 \to Q} F(P_1) = \lim_{P_2 \to Q} F(P_2), \text{ if} \\ Q = (x, y, z) \in \{\text{circular conics}\}. \end{cases}$$

Then we can show that $\tilde{F}(x, y, z)$ is a C^∞-function on R^3 and it satisfies $X = X_{\tilde{F}}$. Q.E.D.

By the above proposition, we have $(\mathscr{L}' \cap \mathscr{I})/(\mathscr{L}' \cap \mathscr{H}) \cong R$. Combining this result with Lemma 3 (ii), we finally obtain

Theorem. $H^1_{\text{Poisson}}(\mathfrak{sl}(2, R)^*)/R \cong \mathscr{C}'$.

The details of the proof and further results concerning higher order Poisson cohomology will appear elsewhere.

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