Warped Products with Critical Riemannian Metric*)

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1. Introduction. Let (B, g) and $(F, \bar{g})$ be two Riemannian manifolds of dimensions n and p respectively, and let f be a positive smooth function on B. Then the warped product space $M = B \times {}_f F$ is defined by the Riemannian metric $\tilde{g} = \pi^*(g) + (f \circ \pi)^2 \sigma^*(\bar{g})$, where π and σ are the projections of $B \times F$ onto B and F, respectively.

Let $n + p = m$. For a local coordinate system (x^a) $(a = 1,2,\cdots, n)$ of B, the metric tensor g has the components (g_{ab}) and $\bar{g}$ on F has the components $(\bar{g}_{\alpha\beta})$ for a local coordinate system (y^α) $(\alpha = 1,2,\cdots, p)$. Hence the metric tensor $\tilde{g}$ on M has the components

$$(\tilde{g}_{ji}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & f^2 \bar{g}_{\alpha\beta} \end{pmatrix}$$

with respect to the local coordinate system $x^i = (x^a, y^\alpha)$ on M and $i, j = 1,\cdots, m$.

Let ∇_b (resp. $\bar{\nabla}_\alpha$) be the components of the covariant derivative with respect to g (resp. $\bar{g}$) and $\left\{ {a \atop b\ c} \right\}$ $\left(\text{resp. } \overline{\left\{ {\alpha \atop \beta\ \gamma} \right\}}\right)$ the christoffel symbol of B (resp. F). Then the christoffel symbol $\widetilde{\left\{ {i \atop j\ k} \right\}}$ of M are given as follows

$$(1.1)\quad \widetilde{\left\{ {c \atop b\ a} \right\}} = \left\{ {c \atop b\ a} \right\},$$

$$(1.2)\quad \widetilde{\left\{ {\alpha \atop d\ \gamma} \right\}} = \frac{(\nabla_d f)}{f} \delta_\gamma{}^\alpha,$$

$$(1.3)\quad \widetilde{\left\{ {a \atop \delta\ \beta} \right\}} = -f(\nabla_b f) g^{ab} \bar{g}_{\delta\beta},$$

$$(1.4)\quad \widetilde{\left\{ {\gamma \atop \beta\ \alpha} \right\}} = \overline{\left\{ {\gamma \atop \beta\ \alpha} \right\}},$$

and the others are zero.

Let $\tilde{R}$, R, and $\bar{R}$ be the curvature tensor of M, B and F respectively, then we get [2, 3, 4, 5]

$$(1.5)\quad \tilde{R}_{dcb}{}^a = R_{dcb}{}^a$$

$$(1.6)\quad \tilde{R}_{d\gamma b}{}^\alpha = \frac{1}{f} (\nabla_d f_b) \delta_\gamma{}^\alpha$$

$$(1.7)\quad \tilde{R}_{\delta\gamma\beta}{}^\alpha = \bar{R}_{\delta\gamma\beta}{}^\alpha - \| f_e \|^2 (\delta_\delta{}^\alpha \bar{g}_{\gamma\beta} - \delta_\gamma{}^\alpha \bar{g}_{\delta\beta})$$

and the others are zero, where $f_b = \nabla_b f$.

The components of Ricci tensors are given by

$$(1.8)\quad \tilde{S}_{cb} = S_{cb} - \frac{p}{f} (\nabla_c f_b),$$

$$(1.9)\quad \tilde{S}_{c\beta} = 0,$$

$$(1.10)\quad \tilde{S}_{\gamma\beta} = \bar{S}_{\gamma\beta} - (p-1) \| f_e \|^2 \bar{g}_{\gamma\beta} - f\Delta f \bar{g}_{\gamma\beta},$$

where Δf is the Laplacian of f for g and $\tilde{S}$, S and $\bar{S}$ are the Ricci tensors of M, B and F respectively.

Let $\tilde{\gamma}$, γ and $\bar{\gamma}$ be the scalar curvatures of M, B and F respectively, then we have

$$(1.11)\quad \tilde{\gamma} = \gamma + f^{-2}\bar{\gamma} - 2pf^{-1}\Delta f - p(p-1) f^{-2} \| f_e \|^2.$$

2. Critical Riemannian metrics. Let $(M = B \times {}_f F, \tilde{g})$ be a compact oriented Riemannian manifold. Consider the following Riemannian functional

$$H(\tilde{g}) = \int_M \tilde{\gamma}^2 \, d\mu,$$

where $d\mu$ is the volume element measured by $\tilde{g}$. A critical point of $H(\tilde{g})$ is called a critical Riemannian metric on M. In particular, every Einstein metric is a critical metric for H on M.

M. Berger [1] obtained the equation of the critical Riemannian metric in the following form in the tensor notations

$$(2.1)\qquad H_{ji} = c\, \tilde{g}_{ji},$$

where c is undetermined constant and H_{ji} is given by

$$(2.2)\quad H_{ji} = 2\tilde{\nabla}_j \tilde{\nabla}_i \tilde{\gamma} - (\tilde{\Delta}\tilde{\gamma}) \tilde{g}_{ji} - 2\tilde{\gamma} \tilde{S}_{ji} + \frac{1}{2} \tilde{\gamma}^2 \tilde{g}_{ji},$$

where $\tilde{\nabla}$ means covariant differentiation with respect to $\tilde{g}$ and $\tilde{\Delta}\tilde{\gamma}$ is the Laplacian of $\tilde{\gamma}$ for $\tilde{g}$.

If the Riemannian metric $\tilde{g}$ on M is a critical Riemannian metric, then the undetermined constant c is determined as

$$(2.3)\quad c = 2\left(\frac{1}{m} - 1\right)\tilde{\Delta}\tilde{\gamma} + \left(\frac{1}{2} - \frac{2}{m}\right) \tilde{\gamma}^2.$$

Hence, by use of (2.2) and (2.3), we have

Lemma 2.1. The Riemannian metric $\tilde{g}$ on warped product space $M = B \times {}_f F$ is critical Riemannian metric if and only if

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(2.4) $m\,\tilde{\nabla}_j\tilde{\nabla}_i\tilde{\gamma} - m\tilde{\gamma}\,\tilde{S}_{ji} - (\tilde{\Delta}\tilde{\gamma})\,\tilde{g}_{ji} + \tilde{\gamma}^2\,\tilde{g}_{ji} = 0.$

3. Main results. On $M = B \times_f F$, by use of (1.1)-(1.4), we get

(3.1) $\tilde{\nabla}_b\tilde{\nabla}_e\gamma = \nabla_b\nabla_e\gamma,$

(3.2) $\tilde{\nabla}_b\tilde{\nabla}_\beta\bar{\gamma} = -\frac{1}{f}\,f_b(\partial_\beta\bar{\gamma}),$

(3.3) $\tilde{\nabla}_\beta\tilde{\nabla}_\alpha\bar{\gamma} = \nabla_\beta\nabla_\alpha\bar{\gamma},$

(3.4) $\tilde{\nabla}_\beta\tilde{\nabla}_e\gamma = 0,$

(3.5) $\tilde{\nabla}_\beta\tilde{\nabla}_e\bar{\gamma} = 0.$

Let M, B, F be compact orientable C^∞ manifolds such that $M = B \times_f F$, and assume that the metric $\tilde{g}$ on M is critical Riemannian metric. Then, from the equation (1.11) and (2.4), we obtain

(3.6) $m\tilde{\nabla}_b\tilde{\nabla}_\beta\{\gamma + f^{-2}\,\bar{\gamma} - 2pf^{-1}\Delta f - p(p-1)\,f^{-2}\,\|f_e\|^2\} = 0.$

From this and (3.1)-(3.5), we get $f_b(\partial_\beta\,\bar{\gamma}) = 0$.

Since $f_b = 0$ means that the function f is constant on M, we have

Theorem 3.1. Let M, B, F be compact orientable C^∞ manifolds such that $M = B \times_f F$. If $\tilde{g}$ on M is critical Riemannian metric, then the warped product space M is the Riemannian product space or $\bar{\gamma}$ is constant on M.

We now assume that the warped product space M is not Riemannian products (in this case, we call M a proper warped product space). Then $\bar{\gamma}$ is constant on M.

Therefore, we see that $\bar{g}$ on F is critical Riemannian metric if and only if

(3.7) $p\bar{\gamma}\,\bar{S}_{\beta\alpha} - \bar{\gamma}^2\,\bar{g}_{\beta\alpha} = 0.$

Hence, if we consider the case

(I) $\bar{\gamma} \equiv 0$ and

(II) $\bar{\gamma}$ is non-zero constant, then we can state

(I) If the scalar curvature $\bar{\gamma}$ on F is zero, then the metric $\bar{g}$ on F is critical Riemannian metric by means of (3.7).

(II) If $\bar{\gamma}$ on F is non-zero constant and $\bar{g}$ on F is critical Riemannian metric, then from (3.7), we see that F is Einstein.

Since the Einstein metric is critical Riemannian metric, we have

Theorem 3.2. Let $M = B \times_f F$ be a proper warped product space with a critical Riemannian metric and $\bar{\gamma} \neq 0$. Then $\bar{g}$ on F is a critical Riemannian metric if and only if F is Einstein.

References

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