The Diophantine Equation $a^x + b^y = c^z$. II

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§1. Introduction. In the previous paper [8], we proposed the following:

Conjecture. *If a, b, c, p, q, r are fixed positive integers satisfying $a^p + b^q = c^r$ with $p, q, r \geq 2$ and $(a, b) = 1$, then the Diophantine equation*

(1) $$a^x + b^y = c^z$$

has the only positive integral solution $(x, y, z) = (p, q, r)$.

When $(p, q, r) = (2,2,2)$, the above Conjecture is called Jeśmanowicz's conjecture. It has been verified that this conjecture holds for many Pythagorean numbers (cf. Jeśmanowicz [3], Takakuwa and Asaeda [5], [6], Takakuwa [7], Adachi [1]).

In [8], we considered the above Conjecture when $(p, q, r) = (2,2,3)$ and showed that it holds for certain a, b, c satisfying $a^2 + b^2 = c^3$.

In this paper, we consider the case $(p, q, r) = (2,2,5)$. Using an argument similar to the one used in [8], we shall prove that the above Conjecture also holds for certain a, b, c satisfying $a^2 + b^2 = c^5$ as specified in Theorem in §2. We shall also give some examples of a, b, c satisfying the conditions of Theorem.

§2. Theorem. We first prepare some lemmas.

In the same way as in the proof of Lemma 1 in [8], we obtain the following:

Lemma 1. *The integral solutions of the equation $a^2 + b^2 = c^5$ with $(a, b) = 1$ are given by*

$$a = \pm u(u^4 - 10u^2v^2 + 5v^4),$$
$$b = \pm v(5u^4 - 10u^2v^2 + v^4),\ c = u^2 + v^2,$$

where u, v are integers such that $(u, v) = 1$ and $u \not\equiv v \pmod 2$.

In the following, we consider the case $u = m, v = 1$; i.e.

(2) $$a = m(m^4 - 10m^2 + 5),\quad b = 5m^4 - 10m^2 + 1,\ c = m^2 + 1$$

and

$$m \text{ is even.}$$

Lemma 2. *Let a, b, c be positive integers satisfying (2). If the Diophantine equation (1) has positive integral solutions (x, y, z), then x and y are even.*

Proof. It suffices to show that $\left(\frac{a}{b}\right) = -1$, $\left(\frac{c}{b}\right) = 1$, $\left(\frac{b}{a'}\right) = -1$ and $\left(\frac{c}{a'}\right) = 1$ with $a = ma'$, where $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol. These imply that x and y are even.

Since $b \equiv 1 \pmod 8$, we have $\left(\frac{m}{b}\right) = 1$. In fact, putting $m = 2^s t$($s \geq 1$ and t is odd), $\left(\frac{m}{b}\right) = \left(\frac{2^s}{b}\right)\left(\frac{t}{b}\right) = \left(\frac{t}{b}\right) = \left(\frac{b}{t}\right) = \left(\frac{1}{t}\right) = 1$.

Hence we have $\left(\frac{a}{b}\right) = \left(\frac{m}{b}\right)\left(\frac{a'}{b}\right) = \left(\frac{a'}{b}\right) = \left(\frac{b}{a'}\right) = \left(\frac{5m^4 - 10m^2 + 1}{m^4 - 10m^2 + 5}\right) = \left(\frac{2}{m^4 - 10m^2 + 5}\right)\left(\frac{5m^2 - 3}{m^4 - 10m^2 + 5}\right) = (-1)\cdot\left(\frac{m^4 - 10m^2 + 5}{5m^2 - 3}\right) = (-1)\cdot 1 = -1$. Thus we obtain $\left(\frac{a}{b}\right) = \left(\frac{b}{a'}\right) = -1$.

We also have $\left(\frac{c}{b}\right) = \left(\frac{b}{c}\right) = \left(\frac{16}{m^2 + 1}\right) = 1$, and $\left(\frac{c}{a'}\right) = \left(\frac{a'}{c}\right) = \left(\frac{16}{m^2 + 1}\right) = 1$. Q.E.D.

Lemma 3. *Let a, b, c be positive integers satisfying $a^2 + b^2 = c^5$ and $(a, b) = 1$. Suppose that there is an odd prime l such that $ab \equiv 0 \pmod l$ and $e \equiv 0 \pmod 5$, where e is the order of c modulo l. If the Diophantine equation* (1) *has positive integral solutions (x, y, z), then $z \equiv 0 \pmod 5$.*

Proof. We may suppose that $b \equiv 0 \pmod l$ without loss of generality.

It follows from $a^2 + b^2 = c^5$ that $a^2 \equiv c^5 \pmod l$. By (1), we see that $a^x \equiv c^z \pmod l$, so $c^{2z} \equiv a^{2x} \equiv c^{5x} \pmod l$. Hence we have $c^{5x-2z} \equiv 1 \pmod l$, which implies $5x - 2z \equiv 0 \pmod e$. Therefore we have $z \equiv 0 \pmod 5$. Q.E.D.

Lemma 4. (a) (Lebesgue [4]). *The Diophan-*

tine equation $x^2 + 1 = y^n$ *has no positive integral solutions* x, y, n *with* $n \geq 2$.

(b) (Cohn [2]). *The Diophantine equation* $x^2 - 20y^4 = 1$ *has the only positive integral solution* $(x, y) = (161,6)$.

We use Lemma 4 to show the following:

Lemma 5. *Let* a, b, c *be positive integers satisfying* (2) *and let* b *be prime. Then the Diophantine equation*

$$a^{2X} + b^{2Y} = c^{5Z}$$

has the only positive integral solution $(X, Y, Z) = (1,1,1)$.

Proof. It follows from Lemma 1 that we have

$a^X = \pm u(u^4 - 10u^2v^2 + 5v^4)$,

$b^Y = \pm v(5u^4 - 10u^2v^2 + v^4)$, $c^Z = u^2 + v^2$,

where $(u, v) = 1$, u is even and v is odd, since b is odd.

Since b is prime, we see that

(3) $v = \pm 1,\ 5u^4 - 10u^2v^2 + v^4 = \pm b^Y$,

or

(4) $v = \pm b^Y,\ 5u^4 - 10u^2v^2 + v^4 = \pm 1$.

We first consider (3). Then we have

$$u^2 + 1 = c^Z,$$

which has the only solution $Z = 1$ from Lemma 4.(a). Thus since $c = m^2 + 1$, we have $u = \pm m$, so $Y = 1$, $X = 1$.

We next consider (4). Then we have

(5) $(v^2 - 5u^2)^2 - 20u^4 = \pm 1$.

The $-$ sign must be rejected since $(v^2 - 5u^2)^2 \equiv -1 \pmod 4$ is impossible. The equation (5) has no non-trivial solutions from Lemma 4.(b). Q.E.D.

Combining Lemmas 2, 3 with Lemma 5, we obtain the following:

Theorem. *Let* $a = \pm m(m^4 - 10m^2 + 5)$, $b = 5m^4 - 10m^2 + 1$, $c = m^2 + 1$ *with* m *even and let* b *be prime. Suppose that there is an odd prime* l *such that* $ab \equiv 0 \pmod l$ *and* $e \equiv 0 \pmod 5$, *where* e *is the order of* c *modulo* l. *Then the Diophantine equation* $a^x + b^y = c^z$ *has the only positive integral solution* $(x, y, z) = (2,2,5)$.

The following table gives some examples of $m\ (\leq 50)$, a, b, c, l, e satisfying the conditions of Theorem.

Table

m	a	b	c	l	e
2	38	41	5	41	20
6	5646	6121	37	6121	3060
8	27688	19841	65	3461	1730
12	231612	102241	145	102241	51120
16	1007696	325121	257	62981	31490
18	1831338	521641	325	521641	260820
20	3120100	796001	401	796001	398000
22	5047262	1166441	485	61	5
26	11705746	2278121	677	41	20
46	204989846	22366121	2117	22366121	11183060

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