# 22. *Kazhdan-Lusztig-Type Multiplicity Formula for Symmetrizable Generalized Kac-Moody Algebras*

By Satoshi NAITO

Department of Mathematics, Shizuoka University

(Communicated by Kiyosi ITÔ, M. J. A., April 12, 1994)

The main purpose of this paper is to generalize the Kazhdan-Lusztig multiplicity formula, proved by Kashiwara and Tanisaki [5], [6] or by Casian [2] for symmetrizable Kac-Moody algebras, to the case of symmetrizable generalized Kac-Moody algebras under a certain restriction (Theorem 4.3). Here a *generalized Kac-Moody algebra* (GKM algebra for short) is a complex contragredient Lie algebra $\mathfrak{g}(A)$ associated to a real square matrix (called a GGCM) $A=(a_{ij})_{i,j\in I}$ indexed by a finite set $I=\{1,2,\ldots,n\}$ satisfying: (1) either $a_{ii}=2$ or $a_{ii}\leq 0$ for $i\in I$; (2) $a_{ij}\leq 0$ if $i\neq j$, and $a_{ij}\in\mathbf{Z}$ for $j\neq i$ if $a_{ii}=2$; (3) $a_{ij}=0$ implies $a_{ji}=0$. This definition of GKM algebras is due to Kac [3, Ch. 11], and is somewhat different from the one by Borcherds [1].

**1. GKM algebras.** For a symmetrizable GGCM $A=(a_{ij})_{i,j\in I}$, there exists a triple $(\mathfrak{h},\Pi,\Pi^\vee)$, called a *realization* of $A$, where $\mathfrak{h}$ is a vector space over $\mathbf{C}$ of dimension $2n-\operatorname{rank}A$, $\Pi=\{\alpha_i\}_{i\in I}\subset\mathfrak{h}^*:=\operatorname{Hom}_{\mathbf{C}}(\mathfrak{h},\mathbf{C})$ and $\Pi^\vee=\{\alpha_i^\vee\}_{i\in I}\subset\mathfrak{h}$ are both linearly independent indexed subsets satisfying $\langle\alpha_j,\alpha_i^\vee\rangle=a_{ij}$ for $i,j\in I$. Here $\langle\cdot,\cdot\rangle$ denotes a duality pairing. The GKM algebra $\mathfrak{g}(A)$ associated to $A$ is the Lie algebra over $\mathbf{C}$ generated by the above vector space $\mathfrak{h}$ and the elements $e_i, f_i$ $(i\in I)$ with the fundamental relations:

$$\text{(F1)}\quad\begin{cases}[h,h']=0 & \text{for } h,h'\in\mathfrak{h},\\ [h,e_i]=\langle\alpha_i,h\rangle e_i,\ [h,f_i]=-\langle\alpha_i,h\rangle f_i & \text{for } h\in\mathfrak{h},\ i\in I,\\ [e_i,f_j]=\delta_{ij}\alpha_i^\vee & \text{for } i,j\in I,\end{cases}$$

$$\text{(F2)}\quad (\operatorname{ad}e_i)^{1-a_{ij}}e_j=0,\ (\operatorname{ad}f_i)^{1-a_{ij}}f_j=0\quad\text{if } a_{ii}=2 \text{ and } j\neq i,$$

$$\text{(F3)}\quad [e_i,e_j]=0,\ [f_i,f_j]=0\quad \text{if } a_{ii},a_{jj}\leq 0 \text{ and } a_{ij}=0.$$

Then we have the *root space decomposition* of $\mathfrak{g}(A)$ with respect to the *Cartan subalgebra* $\mathfrak{h}$: $\mathfrak{g}(A)=\mathfrak{h}\oplus\sum^{\oplus}_{\alpha\in\Delta_+}\mathfrak{g}_\alpha\oplus\sum^{\oplus}_{\alpha\in\Delta_-}\mathfrak{g}_\alpha$, where $\Delta_+\subset Q_+:=\sum_{i\in I}\mathbf{Z}_{\geq 0}\alpha_i$ is the set of *positive roots*, $\Delta_-=-\Delta_+$ is the set of *negative roots*, and $\mathfrak{g}_\alpha$ is the *root space* corresponding to a *root* $\alpha\in\Delta=\Delta_+\cup\Delta_-\subset\mathfrak{h}^*$. Note that $\mathfrak{g}_{\alpha_i}=\mathbf{C}e_i$, $\mathfrak{g}_{-\alpha_i}=\mathbf{C}f_i$ for $i\in I$.

We put $I^{re}:=\{i\in I\mid a_{ii}=2\}$, $I^{im}:=\{i\in I\mid a_{ii}\leq 0\}$, and $\Pi^{re}:=\{\alpha_i\in\Pi\mid i\in I^{re}\}$ the set of *real simple roots*, $\Pi^{im}:=\{\alpha_i\in\Pi\mid i\in I^{im}\}$ the set of *imaginary simple roots*. For $\alpha_i,\alpha_j\in\Pi^{im}$, we say that $\alpha_i$ is *perpendicular* to $\alpha_j$ if $a_{ij}=0$. (Remark that an imaginary simple root $\alpha_i\in\Pi^{im}$ is perpendicular to itself if $a_{ii}=0$.) For $\lambda\in\mathfrak{h}^*$ and $\alpha_i\in\Pi^{im}$, we say that $\alpha_i$ is *perpendicular* to $\lambda$ if $\langle\lambda,\alpha_i^\vee\rangle=0$.

Now fix an element $\Lambda\in P_+:=\{\lambda\in\mathfrak{h}^*\mid\langle\lambda,\alpha_i^\vee\rangle\geq 0\ (i\in I)$, and $\langle\lambda,\alpha_i^\vee\rangle$

$\in \mathbf{Z}_{\geq 0}$ if $a_{ii} = 2\}$. Then we define a subset $\mathcal{S}(\Lambda)$ (resp. $\mathcal{A}(\Lambda)$) of $\mathfrak{h}^*$ to be the set of all sums of distinct (resp. not necessarily distinct), pairwise perpendicular, imaginary simple roots perpendicular to $\Lambda$. For an element $\beta = \sum_{i \in I^{im}} k_i \alpha_i \in \mathcal{A} := \mathcal{A}(0)$, we put $\mathrm{ht}(\beta) = \sum_{i \in I^{im}} k_i$.

The *Weyl group* $W$ of $\mathfrak{g}(A)$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by the *simple reflections* $r_i$ $(i \in I^{re})$. For an element $w \in W$, $\ell(w)$ denotes the *length* of $w$. Put $\Delta^{re} := W \cdot \Pi^{re}$ (the set of *real roots*), and $\Delta^{im} := \Delta \setminus \Delta^{re}$ (the set of *imaginary roots*). For a real root $\alpha = w(\alpha_i)$ $(w \in W, \alpha_i \in \Pi^{re})$, we define the reflection $r_\alpha$ of $\mathfrak{h}^*$ with respect to $\alpha$ by: $r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ $(\lambda \in \mathfrak{h}^*)$, where $\alpha^\vee := w(\alpha_i^\vee) \in \mathfrak{h}$ is the *dual real root* of $\alpha$. Note that $r_\alpha = w r_i w^{-1} \in W$.

From now on throughout this paper, we assume that the GGCM $A = (a_{ij})_{i,j \in I}$ is symmetrizable. So there exists a nondegenerate, symmetric, invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$. Note that the restriction of this bilinear form to the Cartan subalgebra $\mathfrak{h}$ is also nondegenerate, so that it induces on $\mathfrak{h}^*$ a nondegenerate, symmetric, $W$-invariant bilinear form, which is denoted again by $(\cdot|\cdot)$.

**2. Basic representation theory of GKM algebras.** We extend the Bruhat ordering on $W$ to the one on $W \times \mathcal{A}$.

**Definition 2.1** (*Bruhat ordering*). Let $w_1, w_2 \in W$. We write $w_1 \leftarrow w_2$ if there exists some $\gamma \in \Delta^{re} \cap \Delta_+$ such that $w_1 = r_\gamma w_2$ and $\ell(w_1) = \ell(w_2) + 1$. Moreover, for $w, w' \in W$, we write $w \geqslant w'$ if $w = w'$ or if there exist $w_1, \ldots, w_t \in W$ such that $w \leftarrow w_1 \leftarrow \cdots \leftarrow w_t \leftarrow w'$.

**Definition 2.2.** For $\beta = \sum_{k \in I^{im}} m_k \alpha_k$, $\beta' = \sum_{k \in I^{im}} m'_k \alpha_k \in \mathcal{A}$, we write $\beta \geqslant \beta'$ if $m_k \geq m'_k$ for all $k \in I^{im}$.

**Definition 2.3.** For $(w, \beta), (w', \beta') \in W \times \mathcal{A}$, we write $(w, \beta) \geqslant (w', \beta')$ if $w \geqslant w'$ and $\beta \geqslant \beta'$.

**2.1.** The *category* $\mathcal{O}$ is the category of all $\mathfrak{g}(A)$-modules $V$ admitting a *weight space decomposition* $V = \sum^{\oplus}_{\tau \in \mathfrak{h}^*} V_\tau$ with finite-dimensional *weight spaces* $V_\tau$ such that the set of all *weights* of $V$ is contained in a finite union of sets of the form $D(\lambda) := \lambda - Q_+$ $(\lambda \in \mathfrak{h}^*)$. Obviously, highest weight $\mathfrak{g}(A)$-modules, such as the *Verma module* $V(\lambda)$ with highest weight $\lambda$ and its irreducible quotient $L(\lambda)$ for $\lambda \in \mathfrak{h}^*$, are in the category $\mathcal{O}$.

For a module $V$ in the category $\mathcal{O}$, we define the *formal character* ch $V$ of $V$ by $\mathrm{ch}\, V := \sum_{\tau \in \mathfrak{h}^*} (\dim_{\mathbf{C}} V_\tau) e(\tau)$, as an element of the algebra $\mathcal{E}$ with basis $e(\tau)$ $(\tau \in \mathfrak{h}^*)$, called *formal exponentials,* introduced in [3, Ch. 9]. Then there exists a unique set $\{a_\mu\}_{\mu \in \mathfrak{h}^*}$ of nonnegative integers such that $\mathrm{ch}\, V = \sum_{\mu \in \mathfrak{h}^*} a_\mu \,\mathrm{ch}\, L(\mu)$ (equality in the algebra $\mathcal{E}$).

**Definition 2.4.** The above integer $a_\mu$ is called the *multiplicity* of $L(\mu)$ in $V$, and is denoted by $[V : L(\mu)]$.

Note that for a module $V \in \mathcal{O}$ and $\mu \in \mathfrak{h}^*$, $[V : L(\mu)] \neq 0$ if and only if $L(\mu)$ is an irreducible subquotient of $V$.

**2.2.** Here we give two module-theoretical results on GKM algebras for the case of $\mathcal{A}(\Lambda)$, which are essentially established in [7] and [8] for the case of $\mathcal{S}(\Lambda)$.

We choose and fix an element $\rho \in \mathfrak{h}^*$ such that $\langle \rho, \alpha_i^\vee \rangle = (1/2)\cdot a_{ii}$ for all $i \in I$, and we shall use the notation $(w, \beta)\circ \Lambda := w(\Lambda + \rho - \beta) - \rho$ for $(w, \beta) \in W \times \mathscr{A}$ and $\Lambda \in P_+$.

**Theorem 2.5.** *Let $\Lambda \in P_+$, $(w, \beta) \in W \times \mathscr{A}(\Lambda)$. Then any irreducible subquotient of the Verma module $V((w, \beta)\circ\Lambda)$ is isomorphic to $L((w', \beta')\circ\Lambda)$ for some $(w', \beta') \in W \times \mathscr{A}(\Lambda)$ with $(w', \beta') \geqslant (w, \beta)$. Conversely, for any $(w', \beta') \in W \times \mathscr{A}(\Lambda)$ with $(w', \beta') \geqslant (w, \beta)$, $L((w', \beta')\circ\Lambda)$ is isomorphic to an irreducible subquotient of $V((w, \beta)\circ\Lambda)$.*

**Theorem 2.6.** *Let $\Lambda \in P_+$, $(w_1, \beta_1)$, $(w_2, \beta_2) \in W \times \mathscr{A}(\Lambda)$. Then*

$$\begin{aligned} & V((w_1, \beta_1)\circ\Lambda) \hookrightarrow V((w_2, \beta_2)\circ\Lambda) \\ \Leftrightarrow\ & (w_1, \beta_1) \geqslant (w_2, \beta_2) \\ \Leftrightarrow\ & [V((w_2, \beta_2)\circ\Lambda) : L((w_1, \beta_1)\circ\Lambda)] \neq 0. \end{aligned}$$

## 3. Two kinds of character sum formulas.

**3.1. Character sum formula for a Verma module and its application.** The following theorem is proved in [4] in the more general setting of symmetrizable contragredient Lie algebras.

**Theorem A** (see [4]). *Let $\mathfrak{g}(A)$ be a symmetrizable* GKM *algebra, and $V(\lambda)$ the Verma module with highest weight $\lambda \in \mathfrak{h}^*$. Then the module $V(\lambda)$ has a $\mathfrak{g}(A)$-module filtration $V(\lambda) \supset V(\lambda)_1 \supset V(\lambda)_2 \supset \cdots$ such that $V(\lambda)/V(\lambda)_1 \cong L(\lambda)$ as a $\mathfrak{g}(A)$-module, and the following equality holds in the algebra $\mathscr{E}$ :*

$$\sum_{i\geq 1} \mathrm{ch}\, V(\lambda)_i = \sum_{\beta\in\Delta_+} \sum_{\substack{l\geq 1 \\ 2(\lambda+\rho|\beta)=l(\beta|\beta)}} \mathrm{ch}\, V(\lambda - l\beta),$$

*where the roots $\beta \in \Delta_+$ are taken with their multiplicities.*

We quote the next lemma from our previous work.

**Lemma B** (cf. [8, Lemma 4.4]). *Assume that the* GGCM $A = (a_{ij})_{i,j\in I}$ *satisfies the condition that $a_{ii} \neq 0$ $(i \in I)$. Let $\mu \in P_+$, $w \in W$, and $\gamma \in \Delta_+$. Then the following are equivalent*:

(1) $2(w(\mu + \rho) \mid \gamma) = m(\gamma \mid \gamma)$ *for some* $m \in \mathbf{Z}_{\geq 1}$,

(2) *we have either* (a) $\gamma \in \Delta^{re}$ *and* $\ell(r_\gamma w) > \ell(w)$, *or* (b) $w^{-1}(\gamma) \in \Pi^{im}$ *and* $(w^{-1}(\gamma) \mid \mu) = 0$.

*Moreover, in case* (a), *we have $r_\gamma w \geqslant w$ and $m = \langle w(\mu + \rho), \gamma^\vee \rangle$. In case* (b), *we have $m = 1$.*

Then we can show the following theorem, by applying Theorem A together with Theorem 2.6 and Lemma B.

**Theorem 3.1.** *Let $\mathfrak{g}(A)$ be a* GKM *algebra associated to a symmetrizable* GGCM $A = (a_{ij})_{i,j\in I}$ *satisfying the condition that $a_{ii} \neq 0$ $(i \in I)$. Let $\Lambda \in P_+$, $w \in W$, and $\beta, \beta' \in \mathscr{A}(\Lambda)$. If $\beta' \geqslant \beta$ and $\mathrm{ht}(\beta') = \mathrm{ht}(\beta) + 1$, then we have $[V((w, \beta)\circ\Lambda) : L((w, \beta')\circ\Lambda)] = 1$.*

**3.2. Character sum formula for a quotient of two Verma modules.** In this subsection, we assume that the GGCM $A = (a_{ij})_{i,j\in I}$ satisfies the condition that $a_{ii} \neq 0$ $(i \in I)$.

Let $\alpha = w(\alpha_j) \in \Delta_+$ with $w \in W$, $\alpha_j \in \Pi^{im}$, and let $\lambda \in \mathfrak{h}^*$ be such that $2(\lambda + \rho \mid \alpha) = (\alpha \mid \alpha)$. Then, arguing as in [9], we obtain an embedding: $V(\lambda - \alpha) \hookrightarrow V(\lambda)$, and furthermore, we can prove

**Theorem 3.2.** *Let $\mathfrak{g}(A)$ be a* GKM *algebra associated to a symmetrizable*

GGCM $A=(a_{ij})_{i,j\in I}$ *satisfying the condition that* $a_{ii}\neq 0$ $(i\in I)$. *Then, with the above notation, the quotient module* $N(\lambda):=V(\lambda)/V(\lambda-\alpha)$ *has a* $\mathfrak{g}(A)$-*module filtration* $N(\lambda)\supset N(\lambda)_1\supset N(\lambda)_2\supset\cdots$ *such that* $N(\lambda)/N(\lambda)_1\cong L(\lambda)$ *as a* $\mathfrak{g}(A)$-*module, and the following equality holds in the algebra* $\mathcal{E}$ :

$$\sum_{i\geq 1}\operatorname{ch}N(\lambda)_i=\sum_{\beta\in\Delta_+}\sum_{\substack{l\geq 1\\ 2(\lambda+\rho|\beta)=l(\beta|\beta)}}\operatorname{ch}V(\lambda-l\beta)-$$
$$-\sum_{\gamma\in\Delta_+}\sum_{\substack{m\geq 1\\ 2(\lambda-\alpha+\rho|\gamma)=m(\gamma|\gamma)}}\operatorname{ch}V(\lambda-\alpha-m\gamma)-c(\lambda)\operatorname{ch}V(\lambda-\alpha),$$

*where* $c(\lambda)\in\mathbf{Z}$, *the roots* $\beta\in\Delta_+$ *and* $\gamma\in\Delta_+$ *are taken with their multiplicities.*

Let $\Lambda\in P_+$, $w\in W$, and $\alpha_j\in\Pi^{im}$ with $(\Lambda\,|\,\alpha_j)=0$. We put $\lambda:=w(\Lambda+\rho)-\rho$, and $\alpha:=w(\alpha_j)\in W\cdot\Pi^{im}\subset\Delta_+$, then $2(\lambda+\rho\,|\,\alpha)=(\alpha\,|\,\alpha)$. From Theorem 3.1, we know that $[V(\lambda):L(\lambda-\alpha)]=1$. On the other hand, we can compute this multiplicity again, this time using Theorem 3.2, to get the following corollary.

**Corollary 3.3.** *Let* $\lambda=w(\Lambda+\rho)-\rho$ *be as above. Then the constant* $c(\lambda)$ *in Theorem* 3.2 *is equal to* 1.

**4. Kazhdan-Lusztig type multiplicity formula.**

**4.1. A reduction to the case of Kac-Moody algebras.**

**Theorem 4.1.** *Let* $\mathfrak{g}(A)$ *be a* GKM *algebra associated to a symmetrizable* GGCM $A=(a_{ij})_{i,j\in I}$. *Let* $\Lambda\in P_+$, *and* $(w,\beta)$, $(z,\beta')\in W\times\mathcal{A}(\Lambda)$. *Then we have*

$$[V((w,\beta)\circ\Lambda):L((z,\beta')\circ\Lambda)]\geq P_{w,z}(1)\cdot P_{\beta,\beta'},$$

*where* $P_{w,z}(q)$ *is the Kazhdan-Lusztig polynomial in* $q$ *for* $W$, *and* $P_{\beta,\beta'}=1$ *if* $\beta'\geqslant\beta$, *and* $=0$ *otherwise. Moreover, the equality holds if* $\beta=\beta'$.

*Sketch of proof.* The Kazhdan-Lusztig multiplicity formula for symmetrizable Kac-Moody algebras states that if $a_{ii}=2$ for all $i\in I$ (in this case $\mathcal{A}(\Lambda)=\{0\}$), then the equality holds for any $w$, $z\in W$ in the theorem. We can apply this celebrated result to the Kac-Moody algebra $\mathfrak{g}(A_{I^{re}})$ associated to the submatrix $A_{I^{re}}:=(a_{ij})_{i,j\in I^{re}}$, a generalized Cartan matrix, of the GGCM $A=(a_{ij})_{i,j\in I}$. Here the Kac-Moody algebra $\mathfrak{g}(A_{I^{re}})$ is embedded into $\mathfrak{g}(A)$ as a canonical subalgebra with Weyl group $W$. Therefore, the theorem can be proved by the same argument as the one for [10, Theorem 3.4].

**4.2. Main result.** In this subsection, we assume that the GGCM $A=(a_{ij})_{i,j\in I}$ satisfies the condition that $a_{ii}\neq 0$ $(i\in I)$. Note that, in this case, the set $\mathcal{A}(\Lambda)$ coincides with the set $\mathcal{B}(\Lambda)$ for $\Lambda\in P_+$.

By double induction on $\ell(z)-\ell(w)$ and $\operatorname{ht}(\beta')-\operatorname{ht}(\beta)$, using Theorem 4.1 as the starting point of the induction, and Theorem 3.2 together with Corollary 3.3 for the induction step, we can prove our main theorem.

**Theorem 4.2.** *Let* $\mathfrak{g}(A)$ *be a* GKM *algebra associated to a symmetrizable* GGCM $A=(a_{ij})_{i,j\in I}$ *satisfying the condition that* $a_{ii}\neq 0$ $(i\in I)$. *Let* $\Lambda\in P_+$, *and* $(w,\beta)$, $(z,\beta')\in W\times\mathcal{A}(\Lambda)$. *Then we have*

$$[V((w,\beta)\circ\Lambda):L((z,\beta')\circ\Lambda)]=P_{w,z}(1)\cdot P_{\beta,\beta'},$$

*where* $P_{\beta,\beta'}$ *is as in Theorem* 4.1.

From Theorem 4.2 together with Theorem 2.5, we have the following

**Theorem 4.3.** *Let* $\mathfrak{g}(A)$ *be a* GKM *algebra associated to a symmetrizable* GGCM $A = (a_{ij})_{i,j\in I}$ *with* $a_{ii} \neq 0$ $(i \in I)$. *Let* $\Lambda \in P_+$. *Then, for* $(w, \beta) \in W \times \mathscr{A}(\Lambda)$, *we have, in the algebra* $\mathscr{E}$,

$$\mathrm{ch}\, V((w, \beta)\circ\Lambda) = \sum_{(z,\beta')\in W\times\mathscr{A}(\Lambda)} P_{w,z}(1)\cdot P_{\beta,\beta'}\cdot \mathrm{ch}\, L((z, \beta')\circ\Lambda).$$

*Equivalently, for* $(w, \beta) \in W \times \mathscr{A}(\Lambda)$, *we have, in the algebra* $\mathscr{E}$,

$$\mathrm{ch}\, L((w, \beta)\circ\Lambda) = \\ = \sum_{(z,\beta')\in W\times\mathscr{A}(\Lambda)} (-1)^{(\ell(z)+\mathrm{ht}(\beta'))-(\ell(w)+\mathrm{ht}(\beta))} Q_{w,z}(1)\cdot P_{\beta,\beta'}\cdot \mathrm{ch}\, V((z, \beta')\circ\Lambda).$$

*Here* $Q_{w,z}(q)$ $(z \in W)$ *are the inverse Kazhdan-Lusztig polynomials in* $q$ *for* $W$ (*see* [6]).

**Remark.** In view of the Weyl-Kac-Borcherds character formula for $L(\Lambda)$ with $\Lambda \in P_+$ (see [1], [3, Ch. 11]), the restriction on the GGCM $A = (a_{ij})_{i,j\in I}$ in Theorems 4.2 and 4.3, that $a_{ii} \neq 0$ $(i \in I)$, seems to be essential.

## References

[1] R. E. Borcherds: Generalized Kac-Moody algebras. J. Algebra, **115**, 501–512 (1988).

[2] L. Casian: Kazhdan-Lusztig multiplicity formulas for Kac-Moody algebras. C. R. Acad. Sci. Paris, **310**, 333–337 (1990).

[3] V. G. Kac: Infinite Dimensional Lie Algebras. 3rd ed., Cambridge Univ. Press, Cambridge (1990).

[4] V. G. Kac and D. A. Kazhdan: Structure of representations with highest weight of infinite-dimensional Lie algebras. Adv. in Math., **34**, 97–108 (1979).

[5] M. Kashiwara: Kazhdan-Lusztig conjecture for a symmetrizable Kac-Moody Lie algebra. Progress in Math., **87**, Birkhäuser, Boston, pp. 407–433 (1990).

[6] M. Kashiwara and T. Tanisaki: Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebra. II. Progress in Math., **92**, Birkhäuser, Boston, pp. 159–195 (1990).

[7] S. Naito: Bernstein-Gelfand-Gelfand resolution for generalized Kac-Moody algebras. Proc. Japan Acad., **69A**, 27–31 (1993).

[8] S. Naito: The strong Bernstein-Gelfand-Gelfand resolution for generalized Kac-Moody algebras. I. Publ. RIMS, Kyoto Univ., **29**, 709–730 (1993).

[9] A. Rocha-Caridi and N. R. Wallach: Highest weight modules over graded Lie algebras: Resolutions, filtrations and character formulas. Trans. Amer. Math. Soc., **277**, 133–162 (1983).

[10] K. Suto: On the Kazhdan-Lusztig conjecture for Kac-Moody algebras. J. Math. Kyoto Univ., **27**, 243–258 (1987).