15. *Commuting Families of Symmetric Differential Operators*

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Introduction. Many commuting families of differential operators or completely integrable quantum systems have been constructed in connection with root systems (cf. [10] and references therein). Such families often have a certain symmetry in coordinates.

The radial parts of invariant differential operators on symmetric spaces give a good example of a commuting family of differential operators (cf. [1]). In this case some parameters take only some discrete values determined by the dimensions of the root spaces for the symmetric spaces.

On the other hand, [12] generalized the example to have holomorphic parameters if the root system is of type A_n. The same generalization was given by [2], [3], [4], [7], [8] in general root systems. If the root system is of classical type, their operators give examples of the commuting families studied in this note (cf. Remark 3 iii)). Namely we shall determine all the families under the assumption of a symmetry in coordinates.

Let W be the Weyl group of type A_{n-1} with $n \geq 3$ or of type B_n with $n \geq 2$ or of type D_n with $n \geq 4$. We identify W with the group of the coordinate transformations

$$(x_1, \ldots, x_n) \mapsto (\varepsilon_1 x_{\sigma(1)}, \ldots, \varepsilon_n x_{\sigma(n)})$$

of $\boldsymbol{R}^n$, where σ are the elements of the n-th symmetric group $\mathfrak{S}_n$ and

$$\begin{cases} \varepsilon_1 = \cdots = \varepsilon_n = 1 & \text{if } W \text{ is of type } A_{n-1}, \\ \varepsilon_1 = \pm 1, \cdots, \varepsilon_n = \pm 1 & \text{if } W \text{ is of type } B_n, \\ \varepsilon_1 = \pm 1, \cdots, \varepsilon_n = \pm 1 \text{ and } \# \{i\,;\, \varepsilon_i = -1\} \text{ is even} & \text{if } W \text{ is of type } D_n. \end{cases}$$

We examine the Laplacian

$$P = -\frac{1}{2} \sum_{1 \leq j \leq n} \frac{\partial^2}{\partial x_j^2} + V(x)$$

on $\boldsymbol{R}^n$ with a W-invariant potential $V(x)$ which has enough W-invariant commuting differential operators. To be precise we assume that there exist W-invariant differential operators $P_1, \ldots, P_n$ with

$$[P_i, P_j] = 0 \quad \text{for} \quad 1 \leq i < j \leq n$$

such that

$$\begin{cases} P = P_2 - \dfrac{1}{2} P_1^2, \\ P_j = \displaystyle\sum_{1 \leq i_1 < \cdots < i_j \leq n} \partial_{i_1} \cdots \partial_{i_j} + R_j \quad \text{with ord } R_j < j \quad \text{for} \quad 1 \leq j \leq n \end{cases}$$

or

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$$\begin{cases} P = -\frac{1}{2}P_1, \\ P_j = \sum_{1\le i_1<\cdots<i_j\le n} \partial_{i_1}^2 \cdots \partial_{i_j}^2 + R_j \text{ with } \operatorname{ord} R_j < 2j \text{ for } 1 \le j \le n \end{cases}$$

or

$$\begin{cases} P = -\frac{1}{2}P_1, \\ P_n = \partial_1 \cdots \partial_n + R_n \text{ with } \operatorname{ord} R_n < n, \\ P_j = \sum_{1\le i_1<\cdots<i_j\le n} \partial_{i_1}^2 \cdots \partial_{i_j}^2 + R_j \text{ with } \operatorname{ord} R_j < 2j \text{ for } 1 \le j \le n-1, \end{cases}$$

if the type of W is A_{n-1} or B_n or D_n, respectively. Here for simplicity we put $\partial_i = \frac{\partial}{\partial x_i}$ and ord R_j are the orders of differential operators R_j.

In this note, we assume that the coefficients of the differential operators are extended to holomorphic functions on a Zariski open subset Ω' of an open connected neighborhood Ω of the origin of the complexification $\boldsymbol{C}^n$ of $\boldsymbol{R}^n$. Namely there exists a non-zero holomorphic function ϕ on Ω with $\Omega' = \{x \in \Omega\,;\, \phi(x) \neq 0\}$.

Determination of the commuting families. The first theorem says that the potential $V(x)$ is only allowed to be a special function.

Theorem 1. *Under the assumption in the introduction, we can conclude*

$$V(x) = \sum_{1\le i<j\le n} u(x_i - x_j) \qquad \text{if } W \text{ is of type } A_{n-1},$$

$$V(x) = \sum_{1\le i<j\le n} (u(x_i - x_j) + u(x_i + x_j)) + \sum_{1\le j\le n} v(x_j) \quad \text{if } W \text{ is of type } B_n,$$

$$V(x) = \sum_{1\le i<j\le n} (u(x_i - x_j) + u(x_i + x_j)) \qquad \text{if } W \text{ is of type } D_n.$$

Here $u(t)$ and $v(t)$ are following functions with complex numbers $C_1, C_2, \ldots$:

If W is of type A_{n-1} with $n \ge 3$,

(1) $$u(t) = C_1\mathcal{P}(t) + C_2.$$

If W is of type B_n with $n \ge 3$,

(2) $$\begin{cases} u(t) = C_1\mathcal{P}(t) + C_2, \\ v(t) = \dfrac{C_3\mathcal{P}(t)^4 + C_4\mathcal{P}(t)^3 + C_5\mathcal{P}(t)^2 + C_6\mathcal{P}(t) + C_7}{\mathcal{P}'(t)^2} \end{cases}$$

or

(3) $$u(t) = C_1t^{-2} + C_2t^2 + C_3 \text{ and } v(t) = C_4t^{-2} + C_5t^2 + C_6$$

or

(4) $$u(t) = C_1 \text{ and } v(t) \text{ is any even function.}$$

If W is of type D_n with $n \ge 4$, then u is (2) *or* (3).

If W is of type B_2 then $(u(t), v(t))$ is (2) *or* (3) *or* (4) *or*

(5) $$\begin{cases} u(t) = \dfrac{C_3\mathcal{P}\left(\frac{t}{2}\right)^4 + C_4\mathcal{P}\left(\frac{t}{2}\right)^3 + C_5\mathcal{P}\left(\frac{t}{2}\right)^2 + C_6\mathcal{P}\left(\frac{t}{2}\right) + C_7}{\mathcal{P}'\left(\frac{t}{2}\right)^2}, \\ v(t) = C_1\mathcal{P}(t) + C_2 \end{cases}$$

or

(6) $$\begin{cases} u(t) = C_1\mathscr{P}(t) + C_2\dfrac{\left(\mathscr{P}\left(\frac{t}{2}\right) - e_3\right)^2}{\mathscr{P}'\left(\frac{t}{2}\right)^2} + C_3, \\ v(t) = C_4\mathscr{P}(t) + \dfrac{C_5}{\mathscr{P}(t) - e_3} + C_6 \end{cases}$$

or

(7) $v(t) = C_1$ *and* $u(t)$ *is any even function.*

In the above theorem, $\mathscr{P}(t)$ is the Weierstrass elliptic function $\mathscr{P}(t \mid 2\omega_1, 2\omega_2)$ with primitive half-periods ω_1 and ω_2 which are allowed to be infinity and e_3 is a complex number satisfying $\mathscr{P}'^2 = 4(\mathscr{P} - e_1)(\mathscr{P} - e_2)(\mathscr{P} - e_3)$ (cf. [14]). In particular

$$\mathscr{P}(t \mid \sqrt{-1}\pi, \infty) = \sinh^{-2}t + \frac{1}{3} \text{ and } \mathscr{P}(t \mid \infty, \infty) = t^{-2}.$$

Then we note that $(u(t), v(t))$ in (2) has 9 complex parameters including the periods.

Theorem 2. i) *If W is of type B_n, the expression of $V(x)$ by u and v is not unique and then we may assume that the coefficient of $\partial_1\partial_2$ of P_2 equals $2u(x_1 - x_2) - 2u(x_1 + x_2)$ without changing the commuting algebra $\boldsymbol{C}[P_1, \ldots, P_n]$.*

ii) *If W is not of type A_{n-1} or if W of type A_{n-1} and* $\operatorname{ord} R_3 < 2$, *then $\boldsymbol{C}[P_1, \ldots, P_n]$ is uniquely determined by u or (u, v).*

iii) *The commuting differential operators $P_1, \ldots, P_n$ exist for P with the potential $V(x)$ defined by u and v of the form* (1), (2), (4), (5), (6) *and* (7) *according to the type of W, where $C_1, \ldots$ are any complex numbers.*

If W is of type A_{n-1}, the commuting differential operators are given by

$$P_k = \sum_{0 \le j \le [\frac{k}{2}]} \frac{1}{2^j j!(k-2j)!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(u(x_1 - x_2)u(x_3 - x_4)\cdots \cdot u(x_{2j-1} - x_{2j})\partial_{2j+1}\partial_{2j+2}\cdots\partial_k)$$

for $k = 1, \ldots, n$ (cf. [10] and [11]).

If W is of type B_n and

$$u(t) = C_5\mathscr{P}(t), \quad v(t) = \sum_{j=1}^{4} C_j\mathscr{P}(t + \omega_j) - \frac{C_0}{2}$$

with complex numbers $C_0, \ldots, C_5$ and $\omega_3 = -(\omega_1 + \omega_2)$ and $\omega_4 = 0$, then the commuting operators are given by

$$P_n(C_0) = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(q_{\{1,\ldots,k\}}\Delta^2_{\{k+1,\ldots,n\}})$$

(cf. [5]), where

$$\Delta_{\{1,\ldots,k\}} = \sum_{0 \le j \le [\frac{k}{2}]} \frac{1}{2^k j!(k-2j)!} \sum_{w \in W(B_k)} \varepsilon(w)w(u(x_1 - x_2)u(x_3 - x_4)\cdots \cdot u(x_{2j-1} - x_{2j})\partial_{2j+1}\partial_{2j+2}\cdots\partial_k),$$

$$q_{\{1,\ldots,k\}} = \sum_{I_1 \amalg \cdots \amalg I_\nu = \{1,\ldots,k\}} T_{I_1}\cdots T_{I_\nu}, \; q_\phi = 1,$$

$$T_{\{1,\ldots,k\}} = (-C_5)^{k-1}\Big(\frac{C_0}{2}T^0_{\{1,\ldots,k\}}(1) - \sum_{j=1}^{4} C_j T^0_{\{1,\ldots,k\}}(\mathscr{P}(t + \omega_j))\Big),$$

$$T^0_{\{1,\dots,k\}}(\psi) = \sum_{I_1 \amalg \cdots \amalg I_\nu = \{1,\dots,k\}} (-1)^{\nu-1}(\nu-1)!S_{I_1}(\psi)\cdots S_{I_\nu}(\psi),$$

$$S_{\{1,\dots,k\}}(\psi) = \sum_{w\in W(B_k)} w(\psi(x_1)\mathcal{P}(x_1-x_2)\mathcal{P}(x_2-x_3)\cdots\mathcal{P}(x_{k-1}-x_k)).$$

Here $W(B_k)$ and $W(D_k)$ are the Weyl groups of type B_k and D_k, respectively, $W(B_k)$ and $W(D_k)$ and $\mathfrak{S}_k$ are realized as groups of coordinate transformations of $\boldsymbol{R}^k$. For $w \in W(B_k)$, $\varepsilon(w) = 1$ if $w \in W(D_k)$ and -1 otherwise, the sums for $I_1,\dots,I_\nu$ run over all the partitions of $\{1,\dots,k\}$, and for a subset I of $\{1,\dots,n\}$, we define $\Delta_I = \sigma(\Delta_{\{1,\dots,k\}})$ etc. by $\sigma \in \mathfrak{S}_n$ and $k = \#I$ with $\sigma(\{1,\dots,k\}) = I$.

Expanding $P_n(C_0)$ into a polynomial function of the parameter C_0, the operators P_j are given by the coefficients of C_0^{n-j} in the expansion. In fact we have $[P_n(C_0), P_n(C_0')] = 0$.

If W is of type D_n, we have only to put $C_1 = C_2 = C_3 = C_4 = 0$ and $P_n = \Delta_{\{1,\dots,n\}}$ in the above definition. See [6] for other cases of type B_2.

Remark 3. i) If (u, v) is of the form (3), P_j do not exist in general and we need operators of higher order (cf. [10]).

ii) If (u, v) is given by (4), then $\boldsymbol{C}[P_1,\dots,P_n]$ equals the totality of $\mathfrak{S}_n$-invariants in $\boldsymbol{C}\left[-\frac{1}{2}\partial_1^2 + v(x_1),\dots,-\frac{1}{2}\partial_n^2 + v(x_n)\right]$.

iii) If $2\omega_1 = \sqrt{-1}\lambda^{-1}\pi$ and $\omega_2 = \infty$ with $\lambda \neq 0$, (2) is reduced to

$$\begin{cases} u(t) = C_1' \sinh^{-2}\lambda t + C_2', \\ v(t) = C_3' \sinh^{-2}\lambda t + C_4' \sinh^{-2} 2\lambda t + C_5' \sinh^2 \lambda t + C_6' \sinh^2 2\lambda t + C_7'. \end{cases}$$

The commuting differential operators studied by Heckman-Opdam correspond to this case with $C_5' = C_6' = 0$. Moreover if $\omega_1 = \omega_2 = \infty$, then (2) is reduced to

$$\begin{cases} u(t) = C_1' t^{-2} + C_2', \\ v(t) = C_3' t^{-2} + C_4' t^2 + C_5' t^4 + C_6' t^6 + C_7'. \end{cases}$$

iv) Some results stated in this note were announced in [13]. The precise statements and arguments will be given in [11], [5] and [6].

v) Replacing ∂_i, x_j, $[\ ,\ ]$ and ord by $\sqrt{-1}p_i$, q_j, the Poisson bracket $\{\ ,\ \}$ and the degree for p, respectively, we have the same statements as in Theorems 1 and 2, and moreover the operators $P_1,\dots,P_n$ give the integrals of the Hamiltonian corresponding to the Laplacian P (cf. [9] for completely integrable classical systems).

References

[1] Harish-Chandra: Representations of semisimple Lie groups. IV. Amer. J. Math., **77**, 743–777 (1955).

[2] G. J. Heckman: Root system and hypergeometric functions. II. Comp. Math., **64**, 353–373 (1987).

[3] ——: An elementary approach to the hypergeometric shift operators of Opdam. Invent. Math., **103**, 341–350 (1991).

[4] G. J. Heckman and E. M. Opdam: Root system and hypergeometric functions. I. Comp. Math., **64**, 329–352 (1987).

[5] T. Oshima: Completely integrable systems with a symmetry in coordinates. UTMS 94–6, Dept. of Mathematical Sciences, Univ. of Tokyo (1994) (preprint).
[6] H. Ochiai and T. Oshima: Commuting families of differential operators invariant under the action of a Weyl group. II (in preparation).
[7] E. M. Opdam: Root system and hypergeometric functions. III. Comp. Math., **67**, 21–49 (1988).
[8] ——: ditto. IV. ibid., **67**, 191–209 (1988).
[9] M. A. Olshanetsky and A. M. Perelomov: Classical integrable finite dimensional systems related to Lie algebras. Phys. Rep., **71**, 313–400 (1981).
[10] ——: Quantum integrable systems related to Lie algebras. ibid., **94**, 313–404 (1983).
[11] T. Oshima and H. Sekiguchi: Commuting families of differential operators invariant under the action of a Weyl group. UTMS 93–43, Dept. of Mathematical Sciences, Univ. of Tokyo (1993) (preprint).
[12] J. Sekiguchi: Zonal spherical functions on some symmetric spaces. Publ. RIMS Kyoto Univ., **12** Suppl., 455–459 (1977).
[13] H. Sekiguchi: Radial parts of Casimir operators on semisimple symmetric spaces. RIMS KôkyûRoku, **816**, 155–168 (1992) (Japanese).
[14] E. T. Whittaker and G. N. Watson: A Course of Modern Analysis. 4th ed., Cambridge University Press (1927).