12. On Contiguity Relations of the Confluent Hypergeometric Systems

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Introduction. This paper concerns the contiguity relations for the confluent hypergeometric systems M_λ (CHG system, for short) defined on the space $Z_{r,n}$ of $r \times n$ complex matrices of maximum rank $r\,(< n)$. As for the definition of the CHG systems and notations employed in this paper, we adopt those of [7].

In [4], we gave a Lie algebra of contiguity operators (see Definition 2.1) in an explicit form. In the present paper, we show that the contiguity operators, obtained in [4], appear in a natural manner in connection with the root space decomposition of the Lie algebra $\mathfrak{gl}_n(\boldsymbol{C})$ with respect to the maximal abelian subalgebra $\mathfrak{h} = LieH_\lambda$.

1. Root space decomposition. Let $H = H_\lambda = J(\lambda_1) \times \cdots \times J(\lambda_l)$ be a maximal abelian subgroup of $GL(n, \boldsymbol{C})$ corresponding to the composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n, where $J(\lambda_k)$ be the Jordan group of size λ_k.

In the following, we often decompose an $n \times n$ matrix X into blocks according to the composition λ as

$$X = (X_{ij})_{1 \le i,j \le l},$$

where X_{ij} is a $\lambda_i \times \lambda_j$ matrix, which will be called (i, j)-block of X.

We denote by $\mathfrak{h}$ the Lie algebra of H, which is given by

$$\mathfrak{h} = \left\{ h = \oplus_{i=1}^{l} h^{(i)}\,;\quad h^{(i)} = \sum_{k=0}^{\lambda_i - 1} h_k^{(i)} \Lambda_{\lambda_i}^k,\ h_k^{(i)} \in \boldsymbol{C} \right\}$$

and is a maximal abelian subalgebra of $\mathfrak{gl}_n = \mathfrak{gl}_n(\boldsymbol{C})$. The dual space of $\mathfrak{h}$ is denoted by $\mathfrak{h}^*$. For any $h \in \mathfrak{h}$, we consider an endmorphism $ad\ h : \mathfrak{gl}_n \to \mathfrak{gl}_n$ defined by

$$(ad\,h)X := [h, X] = hX - Xh.$$

We say that a non zero element $\beta \in \mathfrak{h}^*$ is a *root* for $\mathfrak{h}$ if the vector space

$$\mathfrak{g}_\beta := \{X \in \mathfrak{gl}_n\,;\,(ad\,h - \beta(h))X = 0 \text{ for all } h \in \mathfrak{h}\}$$

is of dimension greater than or equal to 1. The vector space $\mathfrak{g}_\beta$ will be called the *root subspace*. Note that $\mathfrak{g}_0 = \mathfrak{h}$.

Let $\beta_j\ (j = 1, \ldots, l)$ be an element of $\mathfrak{h}^*$ which sends the matrix $\oplus_{k=1}^{l}(\sum_{i=0}^{\lambda_k - 1} h_i^{(k)} \Lambda_{\lambda_k}^i)$ to the common diagonal element $h_0^{(j)}$ of (j, j)-block. We see that the set Δ of non zero roots for $\mathfrak{h}$ is given by

$$\Delta = \{\beta_i - \beta_j\,;\, i, j = 1, \ldots, l,\ i \neq j\}.$$

Proposition 1.1. *For any root* $\beta_i - \beta_j \in \Delta$,

$$\mathfrak{g}_{\beta_i - \beta_j} = \boldsymbol{C} X_{\beta_i - \beta_j},$$

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where $X_{\beta_i-\beta_j} \in \mathfrak{gl}_n$ *is a matrix element whose only non zero entry locates at the position of the first row and the last column of* (i, j)*-block.*

We define a linear subspace $\mathfrak{g}$ of $\mathfrak{gl}_n$ by

$$\mathfrak{g} = \mathfrak{h} + \sum_{\beta\in\Delta} \mathfrak{g}_\beta \text{ (direct sum).}$$

Note that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_n$.

We study the relation among the generators of $\mathfrak{g}$. Let $H_m^{(i)}$ be an element of $\mathfrak{gl}_n$ such that the (i, i)-block is $\Lambda_{\lambda_i}^m$ and the others are zero matrices. Then

$$\mathfrak{h} = \sum_{i=1}^{l} \sum_{m=0}^{\lambda_i-1} CH_m^{(i)} \text{ (direct sum)}$$

and $\mathfrak{g}$ is spanned by

$$B := \{H_m^{(i)}, X_{\beta_i-\beta_j}; i, j = 1,\dots,l, m = 0,\dots,\lambda_i - 1\}$$

over $\boldsymbol{C}$.

Proposition 1.2. *The elements of* B *satisfy the commutation relations*:

$$[h, X_\beta] = \beta(h)X_\beta, \qquad \text{for } h \in \mathfrak{h}, \beta \in \Delta$$
$$[X_{\beta_i-\beta_j}, X_{\beta_j-\beta_k}] = \delta_{\lambda_j 1}X_{\beta_i-\beta k} \qquad \text{for } i \neq k$$
$$[X_{\beta_i-\beta_j}, X_{\beta_j-\beta_i}] = \delta_{\lambda_j 1}H_{\lambda_i-1}^{(i)} - \delta_{\lambda_i 1}H_{\lambda_j-1}^{(j)}$$
$$[X_\beta, X_\gamma] = 0 \qquad \text{for } \beta, \gamma \in \Delta, \beta + \gamma \notin \Delta \cup \{0\}.$$

2. Contiguity operators and contiguity relations. Let $\mathscr{A}$ be the Weyl algebra on $Z_{r,n}$, i.e. the set of linear differential operators with polynomial coefficients in $z \in Z_{r,n}$ which is equipped with a natural additive structure and the multiplicative structure given by composition as operators.

Let $\mathscr{L}_\lambda(\alpha)$ be the left ideal of $\mathscr{A}$ generated by the operators

$$L_m^{(k)} - \alpha_m^{(k)}, \quad k = 1,\dots,l, m = 0,\dots,\lambda_k - 1,$$
$$M_{ij} + \delta_{ij}, \quad i, j = 0,\dots,\tau - 1,$$
$$\Box_{ij,pq}, \quad i, j = 0,\dots,\tau - 1, p, q = 0,\dots,n - 1$$

which defines the CHG system $M_\lambda(\alpha)$. We denote by $\mathscr{S}(\alpha)$ the sheaf of solutions of the system $M_\lambda(\alpha)$.

Definition 2.1. An element $L \in \mathscr{A}$ is said to be a *contiguity operator* for the system M_λ if there is a $\zeta \in \boldsymbol{Z}^n$ such that L defines a sheaf homomorphism

$$L : \mathscr{S}(\alpha) \to \mathscr{S}(\alpha + \zeta). \tag{2.1}$$

It is to be noted that $L \in \mathscr{A}$ satisfies (2.1) if and only if

$$QL \in \mathscr{L}(\alpha) \quad \text{for any } Q \in \mathscr{L}(\alpha + \zeta).$$

In this section, we change the indexing of entries of an element $z \in Z_{r,n}$ as follows:

$$z = (z^{(1)},\dots,z^{(l)}),$$

where $z^{(k)}$ is a $r \times \lambda_k$ matrix,

$$z^{(k)} = (z_0^{(k)},\dots,z_{\lambda_k-1}^{(k)}), \quad z_j^{(k)} = {}^t(z_{0j}^{(k)},\dots,z_{r-1,j}^{(k)}).$$

Take $X \in \mathfrak{gl}_n$ and let $\{\exp sX\}$ be a 1-parameter subgroup of $GL(n, \boldsymbol{C})$ generated by X. Define a linear differential operator L_X acting on the sheaf of C^∞ function on $Z_{r,n}$ by

$$L_X f(z) := \frac{d}{ds} f(z \exp sX) \big|_{s=0}. \tag{2.2}$$

Now we take $X = X_{\beta_i-\beta_j}$ in (2.2). Noting that

$$\exp sX_\beta = E + sX_\beta \in GL(n, \boldsymbol{C})$$

for $X_\beta \in \mathfrak{g}_\beta$ and sufficiently small $|s|$, we have

$$L_{X_{\beta_i-\beta_j}} = \sum_{k=0}^{r-1} z_{k0}^{(i)} \frac{\partial}{\partial z_{k,\lambda_j-1}^{(j)}}.$$

If we take $X = H_m^{(i)}$ in (2.2) we get $L_{H_m^{(i)}} = L_m^{(i)}$, the operator in the system M_λ. The operators L_X $(X \in B)$ generate a Lie algebra $\tilde{\mathfrak{g}}$ isomorphic to $\mathfrak{g}$.

Let $e_0^{(i)}$ $(i = 1,\dots,l)$ be a unit vector in the parameter space $\boldsymbol{C}^n$ whose non zero component is in the first position of the i-th block.

By using the integral representation of solutions of M_λ ([7], Proposition 1.3), we obtain

Theorem 2.2. *For any root $\beta_i - \beta_j \in \Delta$ $(i \neq j)$, the operator $L_{X_{\beta_i-\beta_j}}$ is a contiguity operator for M_λ such that*

$$L_{X_{\beta_i-\beta_j}} : \mathcal{S}(\alpha) \to \mathcal{S}(\alpha + e_0^{(i)} - e_0^{(j)}).$$

More precisely, for any $\Phi(z;\alpha) \in \mathcal{S}(\alpha)$, we have

$$L_{X_{\beta_i-\beta_j}} \Phi(z;\alpha) = \alpha_{\lambda_j-1}^{(j)} \Phi(z;\alpha + e_0^{(i)} - e_0^{(j)}).$$

Remark 2.3. (i) The differential operator $L_{X_{\beta_i-\beta_j}}$ is just the operator P_{ij} obtained in [4].

(ii) The Lie algebra $\tilde{\mathfrak{g}}$ in this paper is larger than that of [4]. The difference of two Lie algebras comes from the fact that $\tilde{\mathfrak{g}}$ is obtained from that of [4] by adding $L_m^{(i)} (i = 1,\dots,l\,;\, m = 0,\dots,\lambda_i - 2)$.

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