11. *Some Families of Generalized Hypergeometric Functions Associated with the Hardy Space of Analytic Functions*

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Abstract: Recently, several inclusion theorems associated with the Hardy space of analytic functions were proven for various families of generalized hypergeometric functions belonging to one or the other subclasses of the class $\mathscr{A}$ of normalized analytic functions in the open unit disk $\mathcal{U}$. The main objective of this paper is to develop a remarkably simple proof of a unification (and generalization) of many of these inclusion theorems. Some relevant historical remarks and observations are also presented.

1. Introduction and definitions. Let $\mathscr{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk $\mathcal{U}$. Also let $\mathscr{S}$ denote the class of all functions in $\mathscr{A}$ which are *univalent* in $\mathcal{U}$. We denote by $\mathscr{S}^*$ and $\mathscr{K}$ the subclasses of $\mathscr{S}$ consisting of all functions in $\mathscr{A}$ which are, respectively, *starlike* and *convex* in $\mathcal{U}$. Then it follows readily that $f(z) \in \mathscr{K} \Leftrightarrow zf'(z) \in \mathscr{S}$, which indeed is the familiar Alexander theorem (cf., e.g., Duren [3, p.43, Theorem 2.12]). We note also that $\mathscr{K} \subset \mathscr{S}^* \subset \mathscr{S}$.

Let $\mathscr{H}^p (0 < p \leq \infty)$ denote the Hardy space of analytic functions $f(z)$ in $\mathcal{U}$, and define the integral means $M_p(r, f)$ by

$$M_p(r, f) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} & (0 < p < \infty) \\ \sup\limits_{0 \leq \theta < 2\pi} |f(re^{i\theta})| & (p = \infty). \end{cases} \tag{1.2}$$

Definition 1. A function $f(z)$, analytic in $\mathcal{U}$, is said to belong to the Hardy space $\mathscr{H}^p (0 < p \leq \infty)$ if

$$\lim_{r \to 1-} \{M_p(r, f)\} < \infty \quad (0 < p \leq \infty). \tag{1.3}$$

For $1 \leq p \leq \infty$, $\mathscr{H}^p$ is a Banach space with the norm $\| f \|_p$ defined by (cf., e.g., Duren [2, p. 23]; see also Koosis [11])

$$\| f \|_p = \lim_{r \to 1-} \{M_p(r, f)\} \quad (1 \leq p \leq \infty). \tag{1.4}$$

Furthermore, $\mathscr{H}^\infty$ is the familiar class of *bounded* analytic functions in $\mathcal{U}$,

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whereas $\mathcal{H}^2$ is the class of power series $\sum a_n z^n$ with $\sum |a_n|^2 < \infty$.

Definition 2. Let $\lambda_j (j = 1, \ldots, l)$ and $\mu_j (j = 1, \ldots, m)$ be complex numbers such that $\mu_j \neq 0, -1, -2, \cdots (j = 1, \ldots, m)$. Then the generalized hypergeometric function ${}_lF_m(z)$ is defined by

$$
\begin{aligned}
{}_lF_m(z) &\equiv {}_lF_m(\lambda_1, \ldots, \lambda_l; \mu_1, \ldots, \mu_m; z) \\
&= F_m \left[\begin{matrix} \lambda_1, \ldots, \lambda_l; \\ \\ \mu_1, \ldots, \mu_m; \end{matrix} \quad z \right] \\
&= \sum_{n=0}^{\infty} \frac{(\lambda_1)_n \cdots (\lambda_l)_n}{(\mu_1)_n \cdots (\mu_m)_n} \frac{z^n}{n!} \quad (l \leq m + 1),
\end{aligned} \tag{1.5}
$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined, in terms of Gamma functions, by

$$
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbf{N} = \{1, 2, 3, \ldots\}) \end{cases}. \tag{1.6}
$$

We note that the ${}_lF_m$ series in (1.5) converges absolutely for (cf., e.g., Erdélyi *et al.* [4, Chapter 4])

(i) $|z| < \infty$ if $l < m + 1$;

(ii) $z \in \mathcal{U}$ if $l = m + 1$;

(iii) $z \in \partial\mathcal{U} = \{z : z \in \mathbf{C} \text{ and } |z| = 1\}$ if $l < m + 1$;

provided further that

$$
\operatorname{Re}\left(\sum_{j=1}^{m} \mu_j - \sum_{j=1}^{l} \lambda_j\right) > 0,
$$

unless the series terminates.

Inclusion theorems for certain generalized hypergeometric functions, associated with the Hardy space $\mathcal{H}^\infty$, were proven recently by Jung *et al.* [5] by applying a one-parameter family of integral operators, and by Kim *et al.* [7] by making use of a fractional integral operator involving the Gauss hypergeometric function ${}_2F_1$ in the kernel (see also [15, Chapter 25]). Each of these inclusion theorems also involves the subclasses $\mathfrak{R}$ and $\mathfrak{R}(\gamma)$ of the class $\mathcal{A}$, which are given by

Definition 3. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathfrak{R}(\gamma)$ if it satisfies the inequality:

$$
\operatorname{Re}\{f'(z)\} > \gamma \quad (z \in \mathcal{U}; 0 \leq \gamma < 1), \tag{1.7}
$$

so that, obviously,

$$
\mathfrak{R}(\gamma) \subseteq \mathfrak{R} \equiv \mathfrak{R}(0) \quad (0 \leq \gamma < 1). \tag{1.8}
$$

It should be remarked in passing that the class $\mathfrak{R}$ was studied rather systematically by MacGregor [12] who indeed referred to numerous earlier works investigating various properties of functions whose derivative has a positive real part. As a matter of fact, a more general class of functions than those satisfying the inequality $\operatorname{Re}\{f'(z)\} > 0 \quad (z \in \mathcal{U})$ is the class of *close-to-convex* functions considered by Kaplan [6]. (See also Duren [3].)

An interesting unification (and generalization) of the aforementioned inclusion theorems of Jung *et al.* [5] and Kim *et al.* [7] was proven, by means of a remarkably simple technique, by Srivastava [13]. We choose to recall Srivastava's result as

Theorem 1 (Srivastava [13]). *Let the function* $z\,{}_lF_m(\lambda_1,\dots,\lambda_l;\mu_1,\dots,\mu_m;z)$ $(l \leq m+1)$ *be in the class* $\mathfrak{R}(\gamma)$ $(0 \leq \gamma < 1)$. *Suppose also that the function* $\Phi(z)$ *is defined, in terms* of *a generalized hypergeometric function, by*

$$\text{(1.9)}\quad \Phi(z) = z\,{}_{l+s}F_{m+s}\left[\begin{matrix}\lambda_1,\dots,\lambda_l,\ \alpha_1,\dots,\alpha_s;\\ \\ \mu_1,\dots,\mu_m,\ \beta_1,\dots,\beta_s;\end{matrix}\ z\right]\quad (l \leq m+1;\, s \in \boldsymbol{N})$$

for (*real or complex*) *parameters* $\alpha_1,\dots,\alpha_s$ and $\beta_1,\dots,\beta_s$ *such that* $\beta_j \neq 0, -1, -2, \cdots$ $(j=1,\dots,s)$.

Then $\Phi(z) \in \mathcal{H}^\infty$ *and, more precisely,*

(1.10) $|\Phi(z)| < \infty$ $(z \in \bar{\mathcal{U}} = \mathcal{U} \cup \partial\mathcal{U} = \{z : z \in \boldsymbol{C}$ and $|z| \leq 1\})$,

provided that

$$\operatorname{Re}\left(\sum_{j=1}^{s}\beta_j - \sum_{j=1}^{s}\alpha_j\right) > 0.$$

The main object of the present paper is to develop a generalization, analogous to Theorem 1, of the following result which was given elsewhere by Kim and Srivastava [8] as an application of a certain one-parameter *additive* family of operators considered earlier by Komatu ([9]; see also [10]) (and, more recently, by Srivastava and Owa [14]).

Theorem 2 (Kim and Srivatava [8]). *The generalized hypergeometric function* $\Psi(z)$ *defined by*

$$\text{(1.11)}\quad \Psi(z) = z\,{}_{l+s}F_{m+s}\left[\begin{matrix}\lambda_1,\dots,\lambda_l,\ 1,\dots,1;\\ \\ \mu_1,\dots,\mu_m,\ 2,\dots,2;\end{matrix}\ z\right]\quad (l \leq m+1;\, s \in \boldsymbol{N})$$

is in the Hardy space $\mathcal{H}^\infty$ *if*

(1.12) $$z\,{}_lF_m(z) \in S^* \text{ and } s \in \boldsymbol{N}\setminus\{1, 2\}$$

or if

(1.13) $$z\,{}_lF_m(z) \in \mathcal{K} \text{ and } s \in \boldsymbol{N}\setminus\{1\}.$$

2. Inclusion theorem for $\mathcal{S}$ and its subclasses. Our main result (depicting the inclusion property associated with the Hardy space and the classes $\mathcal{S}$, $\mathcal{S}^*$, and $\mathcal{K}$) is contained in

Theorem 3. *Let the parameters* $\alpha_1,\dots,\alpha_s$ *and* $\beta_1,\dots,\beta_s$ *be complex numbers such that*

$$\beta_j \neq 0, -1, -2, \cdots \ (j=1,\dots,s),$$

and let ω *be defined by*

$$\text{(2.1)}\quad \omega = \sum_{j=1}^{s}\beta_j - \sum_{j=1}^{s}\alpha_j.$$

Then the generalized hypergeometric function $\Phi(z)$ *defined by* (1.9) *is in the Hardy space* $\mathcal{H}^\infty$ (*and, more precisely, the assertion* (1.10) *holds true*) *if*

(2.2) $$z\,{}_lF_m(z) \in \mathcal{S} \supset \mathcal{S}^* \supset \mathcal{K} \text{ and } \operatorname{Re}(\omega) > 2$$

or if

(2.3) $$z\,{}_lF_m(z) \in \mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \text{ and } \operatorname{Re}(\omega) > 1.$$

In place of the one-parameter additive family of operators (used elsewhere to prove Thcorem 2), our proof of the general result (Theorem 3) is based largely upon the coefficient inequalities asserted by the following

Lemma (cf. de Branges [1]; see also Duren [3, pp. 44-45]). *Let the function* $f(z)$ *be defined by* (1.1).

Then

(2.4) $$f \in \mathcal{S} \supset \mathcal{S}^* \supset \mathcal{K} \Rightarrow |a_n| \leq n \quad (n \in \boldsymbol{N}^* = \boldsymbol{N} \setminus \{1\})$$

and

(2.5) $$f \in \mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \Rightarrow |a_n| \leq 1 \quad (n \in \boldsymbol{N}^*).$$

Furthermore, strict inequality holds true for all n *unless* f *is a rotation of the Koebe function*

(2.6) $$K_0(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n\, z^n \quad (z \in \mathcal{U})$$

in (2.4), *and of the function*

(2.7) $$L_0(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \quad (z \in \mathcal{U})$$

in (2.5).

Proof of Theorem 3. For the sake of convenience, we write

(2.8) $$\Omega_n = \frac{(\lambda_1)_n \cdots (\lambda_l)_n}{(\mu_1)_n \cdots (\mu_m)_n} \text{ and } \Delta_n = \frac{(\alpha_1)_n \cdots (\alpha_s)_n}{(\beta_1)_n \cdots (\beta_s)_n} \quad (n \in \boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}).$$

Thus, by applying the assertion (2.4) of the Lemma, the hypothesis

$$z\, {}_lF_m \left[\begin{matrix} \lambda_1, \ldots, \lambda_l ; & \\ & z \\ \mu_1, \ldots, \mu_m ; & \end{matrix} \right] = z + \sum_{n=2}^{\infty} \frac{\Omega_{n-1}}{(n-1)!} z^n \in \mathcal{S}$$

implies that

(2.9) $$\frac{|\Omega_{n-1}|}{(n-1)!} \leq n \quad (n \in \boldsymbol{N}^*).$$

Next, from the definition (1.6) and Stirling's asymptoic expansion for the Gamma function (cf., e.g., Erdélyi *et al.* [4, p. 47, Section 1.18]), it is not difficult to show for Δ_n defined by (2.8) and with fixed parameters α_j and β_j $(j = 1, \ldots, s)$ that

(2.10) $$\Delta_n = \Lambda^{-1} n^{-\omega} [1 + O(n^{-1})] \quad (n \to \infty),$$

where, for convenience,

(2.11) $$\Lambda = \frac{\Gamma(\alpha_1) \cdots \Gamma(a_s)}{\Gamma(\beta_1) \cdots \Gamma(\beta_s)}$$

and ω is given, as before, by (2.1).

Now, for the function $\Phi(z)$ defined by (1.9), we have

$$|\Phi(z)| \leq |z| + \sum_{n=2}^{\infty} \frac{|\Omega_{n-1}|}{(n-1)!} |\Lambda_n| |z|^n,$$

which, for $z \in \bar{\mathcal{U}}$, yields

(2.12) $$|\Phi(z)| \leq 1 + \sum_{n=2}^{\infty} |c_n|,$$

where

(2.13) $$|c_n| = \frac{|\Omega_{n-1}|}{(n-1)!} |\Lambda_n| \quad (n \in \boldsymbol{N}^*).$$

Making use of (2.9) and (2.10), we find from (2.13) that

$$(2.14)\qquad |c_n| \leq \frac{M}{|\Lambda|}\frac{1}{n^{\mathrm{Re}(\omega)-1}} \quad (n \geq N \in \boldsymbol{N}\,;\, M > 0),$$

which proves that the power series of the function $\Phi(z)$ converges absolutely for *each* $z \in \bar{\mathcal{U}}$, provided that ω is constrained precisely as in the assertion (2.2).

This evidently completes our remarkably simple (and direct) proof of the assertion (2.2) of Theorem 3. The proof of the assertion (2.3) of Theorem 3 would similarly make use of the following consequence of (2.5):

$$(2.15)\qquad \frac{|\Omega_{n-1}|}{(n-1)!} \leq 1 \quad (n \in \boldsymbol{N}^*)$$

instead of the inequality (2.9).

3. Applications. Each of the assertions (2.2) and (2.3) of Theorem 3 is very general in character. By setting

$$(3.1)\qquad a_j = 1 \text{ and } \beta_j = 2 \quad (j = 1,\ldots,s),$$

and by *further* resticting s so that ω (which, in this *special* case, equals s) is constrained as in the assertions (2.2) and (2.3), we are led at once to a *mild* generalization of Theorem 2, contained in the following

Corollary. *The generalized hypergeometric function* $\Psi(z)$ *defined by* (1.11) *is in the Hardy space* $\mathcal{H}^\infty$ *(and, more precisely, the assertion* (1.10) *holds true when* Φ *is replaced by* Ψ*) if*

$$(3.2)\qquad z\,{}_lF_m(z) \in \mathcal{S} \supset \mathcal{S}^* \supset \mathcal{K} \text{ and } s \in \boldsymbol{N}\setminus\{1,2\}$$

or if

$$(3.3)\qquad z\,{}_lF_m(z) \in \mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \text{ and } s \in \boldsymbol{N}\setminus\{1\}.$$

Many more interesting consequences of Theorem 3 can be deduced by assigning suitable special values to the various parameters involved in the generalized hypergeometric function $\Phi(z)$ defined by (1.9).

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