10. *Principal Transformations between Riemann Surfaces**)

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1. Introduction. Let X be a compact or open Riemann surface, $M(X)$ denote its meromorphic function field and let $\mathrm{Div}(X)$ denote the group of divisors on X. We are interested in the following question: how are the structures of $M(X)$ and $\mathrm{Div}(X)$ reflected to the conformal (or topological) structure of X. In this note, we are concerned with principal transformations between two Riemann surfaces (introduced by Nakai and Sario [12]). In particular, we are interested in the existence and the non-existence of special principal transformations between two Riemann surfaces.

Definition 1.1. Let X and Y be Riemann surfaces. A bijection Φ of X to Y is a *principal transformation* provided that for every divisor $D = \sum_{i=1}^{m} n_i P_i$ on X, D is principal (i.e. the divisor of a meromorphic function) if and only if $\Phi(D) = \sum_{i=1}^{m} n_i \Phi(P_i)$ is a principal divisor on Y.

Of course, if Φ is a principal transformation, so is Φ^{-1}.

Next, we give the definition of special principal transformations. Again, let X and Y be Riemann surfaces. Let σ be an isomorphism of $M(X)$ onto $M(Y)$ as abstract fields. It was proved by Bers [2] that σ induces an automorphism of $\boldsymbol{C}$. For simplicity's sake, in this note, we always assume every isomorphism satisfies $\sigma(\sqrt{-1}) = \sqrt{-1}$. Then, associated to σ we have a bijection Φ of X to Y satisfying $\sigma|_{\boldsymbol{C}}(f \circ \Phi^{-1}(P)) = \sigma(f)(P)$ for every $f \in M(X)$ and $P \in Y$. For a complete proof of this fact, the reader should be referred to [9, 11, 12]. This shows that a divisor $\sum n_i P_i \in \mathrm{Div}(Y)$ is the divisor of $\sigma(f)$ if and only if $\sum n_i \Phi^{-1}(P_i)$ is the divisor of f modulo non-zero holomorphic functions. Hence, Φ^{-1} is a principal transformation and so is Φ.

Definition 1.2. We say that a principal transformation is *special* if it is induced by an abstract field isomorphism of meromorphic function fields of two Riemann surfaces.

Assume X and Y are conformally equivalent. Let φ be a conformal mapping of X onto Y. Then, $\sigma : f \mapsto f \circ \varphi^{-1}$ is an isomorphism of $M(X)$ onto $M(Y)$. Obviously, $\sigma|_{\boldsymbol{C}}(f \circ \varphi^{-1}(P)) = \sigma(f)(P)$ for every $f \in M(X)$ and $P \in Y$. Hence, every conformal mapping of X onto Y is a special principal transformation.

Conversely, if X and Y are open, every special principal transformation is a conformal mapping (Iss'sa[9], Nakai[11], Nakai and Sario[12]). In case X and Y are compact, however, it is not necessary a conformal mapping (Heins[7], Kato[10], Gouma[6]).

In the next section, we shall prove that if there exists a principal

*) Dedicated to Professor Mitsuru Nakai on his 60th birthday.

transformation between two Riemann surfaces, these Riemann surfaces are simultaneously open or compact (Theorem 2.1). In Section 3, we show that there exists a non-special principal transformation between compact Riemann surfaces of genus one (Theorem 3.3). This relates to the closing problem of Nakai and Sario [12]. We are not interested in case of genus zero, because every bijective self-mapping of the Riemann sphere is a principal transformation.

It remains the following question: for compact Riemann surfaces of genus greater than one, does there exist a non-special principal transformation? By Theoem 3.4, if there exists, it should be discontinuous.

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2. A property of principal transformations. In this section, we prove the following:

Theorem 2.1. *Let X and Y be Riemann surfaces. Assume there exists a principal transformation between X and Y. Then, X and Y are simultaneously compact or open. If they are compact, they are topologically equivalent, i.e. they have the same genera.*

Remark 2.2. Although Nakai and Sario [12] stated their assertion in terms of isomorphisms of meromorphic function fields, actually, they proved the above assertion in the context of their proof using the boundedness property of principal transformations. Gouma [6] gave an alternative proof for the assertion of Nakai and Sario. We shall give a proof without appealing to the boundedness property of principal transformations. Our proof is also different from Gouma's one.

Proof. Assume X is open. Let Φ be a principal transformation of X to Y. Let P be an arbitrary point on X. By the Behnke-Stein-Florack theorem, there exists a meromorphic (actually, holomorphic) function f on X whose divisor is P. Since Φ is principal, $\Phi(P)$ is a principal divisor i. e. there exists $F \in M(Y)$ whose divisor is $\Phi(P)$. Evidently F is a nonconstant holomorphic function on Y. Hence, Y is open. Since Φ^{-1} is also principal, *vice versa.*

Assume X and Y are compact of genus $g(X)$ and $g(Y)$, respectively. Let P be an arbitrary point on X. Let $0 < n_1 < n_2 < \cdots < n_j < \cdots$ be the nongaps at P, i.e. the complementary set of the Weierstrass gap sequence at P in N^+. Then, there exists a sequence of functions $\{f_j\}$ in $M(X)$ such that the polar divisor of f_j is n_jP. Since Φ is principal, there exists a sequence $\{F_j\}$ in $M(Y)$ such that the polar divisor of F_j is $n_j\Phi(P)$. Hence, $\{n_j\}$ is a subset of the non-gaps at $\Phi(P)$. Since $g(X) = {}^{\#}(N^+ - \{n_j\})$, we have $g(Y) \leq g(X)$. Applying the same argument to Φ^{-1}, we have $g(X) = g(Y)$.

Remark 2.3. The proof of the second assertion of Theorem 2.1 shows that many conformal invariants, the Weierstrass gap sequence, the gonality, the Clifford index etc. (for the definitions of these terms cf. [1, 3, 4, 5]), are also invariant under principal transformations.

3. Existence of special principal transformations. In this section we are concerned with the existence and the non-existence of special principal transformations of compact Riemann surfaces.

Let X be a compact Riemann surface of genus $g \geq 1$. Let $\omega_1, \ldots, \omega_g$ be a basis of the space of holomorphic differentials on X. Choosing a base point $P_0 \in X$, we have the map u_X of X to the Jacobian variety $J(X) = \boldsymbol{C}^g/\Lambda$ defined by $u_X(P) = \left(\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_g\right)$, where Λ is a lattice generated by the period matrix. For $D = \sum n_i P_i \in \mathrm{Div}(X)$, we set $u_X(D) = \sum n_i u_X(P_i)$. Then, u_X can be extended to the map of $\mathrm{Div}(X)$ to $J(X)$. Abel's theorem states that for divisors $D_1, D_2 \in \mathrm{Div}(X)$ of the same degree, $D_1 - D_2$ is a principal divisor if and only if $u_X(D_1) = u_X(D_2)$. As usual, we denote by $W_d(X)$ the image of all positive divisors of degree d on X under u_X. It is well known that $J(X) = W_g(X)$ and $W_1(X)$ is biholomorphic to X. (cf. Arbarello *et al.*[1], Farkas and Kra[5] etc.)

Lemma 3.1. *Let X and Y be compact Riemann surfaces of the same genus and let Φ be a bijection of X to Y. Then, Φ is a principal transformation if and only if there is an (abstract) isomorphism Φ^* of $J(X)$ to $J(Y)$ satisfying that* $\Phi^* \circ u_X = u_Y \circ \Phi$ and $\Phi^*(W_1(X)) = W_1(Y)$.

Remark 3.2. We have to be careful to use the term "isomorphism." In this lemma, Φ^* is an isomorphism of abstract abelian groups.

Proof. Assume that Φ is a principal transformation. Let $D_1, D_2 \in \mathrm{Div}(X)$ be divisors of the same degree such that $u_X(D_1) = u_X(D_2)$. Then, by Abel's theorem, $D_1 - D_2$ be a principal divisor on X. Since Φ is a principal transformation, $\Phi(D_1 - D_2) = \Phi(D_1) - \Phi(D_2)$ is a principal divisor on Y. Again, by Abel's theorem, $u_Y(\Phi(D_1)) = u_Y(\Phi(D_2))$. Hence, there is a well-defined map Φ^* satisfying $\Phi^* \circ u_X = u_Y \circ \Phi$. It is evident by construction that Φ^* is an isomorphism and $\Phi^*(W_1(X)) = W_1(Y)$.

The converse is similar. This completes the proof.

If X is of genus one, then $J(X) = W_1(X)$ and we may assume that Λ is generated by $\{1, \tau\}$ for some $\tau \in \boldsymbol{C}$ in the upper half plane (i.e. $\mathrm{Im}\ \tau > 0$). Let Y be another Riemann surface of genus one. Assume the lattice associated to $J(Y)$ is generated by $\{1, \rho\}$, with $\mathrm{Im}\ \rho > 0$. Set $f(z) = az + b\bar{z}$, where $a = \dfrac{\rho - \bar{\tau}}{\tau - \bar{\tau}}$, $b = \dfrac{\tau - \rho}{\tau - \bar{\tau}}$. Then, it is easy to see that $f(z)$ is an isomorphism of $J(X)$ to $J(Y)$. Noting that u_X and u_Y are biholomorphic, by Lemma 3.1, we have $u_Y^{-1} \circ f \circ u_X$ is a principal transformation of X to Y.

Hence, for every pair of compact Riemann surfaces of genus one, there is a principal transformation between them.

On the other hand, there are infinitely many pairs of compact Riemann surfaces X, Y of genus one such that $M(X)$ is not isomorphic to $M(Y)$ ([6, 10, 12]). Hence, we have:

Theorem 3.3. *For infinitely many pairs of compact Riemann surfaces X, Y of genus one, there exist non-special principal transformations.*

Non-special principal transformations constructed above are continuous.

Contrary to the case of genus one, in case of genus greater than one we have:

Theorem 3.4. *Let X and Y be compact Riemann surfaces of genus $g > 1$. If there exists a continuous principal transformation Φ of X to Y, then X and Y are conformally equivalent and Φ is special.*

Proof. It is obvious that Φ^* constructed in the proof of Lemma 3.1 is also continuous. Since $J(X)$ and $J(Y)$ are Lie groups, every continuous isomorphism of $J(X)$ to $J(Y)$ is real analytic [8]. Hence, Φ^* is a real analytic isomorphism of $J(X)$ onto $J(Y)$, so it can be extended to a real analytic automorphism of $\boldsymbol{C}^g$. Therefore, Φ^* is a linear transformation of $\boldsymbol{R}^{2g}$ to itself. If Φ^* were not holomorphic, $W_1(Y)$ would not be an analytic variety. This is a contradiction. Hence, Φ^* is holomorphic and so is Φ. Then, it is obvious that X and Y are conformally equivalent and Φ is induced by an isomorphism of $M(X)$ to $M(Y)$.

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