74. *On Jacobi-Perron Algorithm*

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§1. Introduction. Let $X = (0,1)^2 \ni (\alpha, \beta)$ and assume that $1, \alpha, \beta$ are linearly independent over $\boldsymbol{Q}$. Put

$$a(\alpha, \beta) := \left[\frac{1}{\alpha}\right], \; b(\alpha, \beta) := \left[\frac{\beta}{\alpha}\right]$$

and

$$T(\alpha, \beta) := \left(\frac{\beta}{\alpha} - b(\alpha, \beta), \frac{1}{\alpha} - a(\alpha, \beta)\right),$$

then T is a transformation of X into itself. $(X, T, a(\alpha, \beta), b(\alpha, \beta))$ is called *Jacobi-Perron algorithm.* Put

$$(a_n, b_n) := (a(T^{n-1}(\alpha, \beta)), b(T^{n-1}(\alpha, \beta))) \;\; n = 1,2,\cdots,$$

$$A(a, b) := \begin{pmatrix} a & 0 & 1 \\ 1 & 0 & 0 \\ b & 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} q_n & q_{n-2} & q_{n-1} \\ p_n & p_{n-2} & p_{n-1} \\ r_n & r_{n-2} & r_{n-1} \end{pmatrix} := A(a_1, b_1) \cdots A(a_n, b_n) \;\; n = 1,2,\cdots,$$

then it is shown that

$$\max(|\alpha - p_n/q_n|, |\beta - r_n/q_n|) \to 0 \quad \text{as } n \to \infty.$$

$(p_n/q_n, r_n/q_n)$ is called the n-th *convergent* of (α, β). If

$$T^m(\alpha, \beta) = T^n(\alpha, \beta) \quad \text{for } m \neq n,$$

then (α, β) is called *periodic.*

Suppose now (α, β) is periodic and let

$$T^m(\alpha, \beta) = T^{m+p}(\alpha, \beta) \quad p \geqq 1.$$

Put

$$M = A(a_{m+1}, b_{m+1}) \cdots A(a_{m+p}, b_{m+p}).$$

M is a 3×3 integral matrix. Perron [1] proved the following result.

The following conditions (1), (2) *are equivalent.*

(1) *M has the eigenvalues $\lambda, \lambda_1, \lambda_2$ such that*

$$\begin{cases} \lambda \in \boldsymbol{R} & \lambda > 1 \\ \lambda_1, \lambda_2 : \text{imaginary} & |\lambda_1| = |\lambda_2| < 1 \end{cases}$$

and the column vector ${}^t(1, \alpha, \beta)$ is the eigenvector for λ.

(2) *The order of approximations of (α, β) by the convergents $(p_n/q_n, r_n/q_n)$ is of exponent $1/2$, i.e. for some $K > 0$*

$$\sqrt{q_n}\, |q_n\alpha - p_n| < K, \quad \sqrt{q_n}\, |q_n\beta - r_n| < K \quad \text{for all } n \in \boldsymbol{N}.$$

It is not difficult to see that if (α, β) is periodic and the condition (1) is satisfied, then for some K

$$(*)\quad \begin{array}{l} |q_np_{n-1}-q_{n-1}p_n|<K\sqrt{q_n}, \qquad |q_np_{n-2}-q_{n-2}p_n|<K\sqrt{q_n}, \\ |q_nr_{n-1}-q_{n-1}r_n|<K\sqrt{q_n} \text{ and } |q_nr_{n-2}-q_{n-2}r_n|<K\sqrt{q_n}. \end{array}$$

The purpose of the paper is to show

Theorem. *Let* $N=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ *be an integral matrix and* λ *be a real eigenvalue of* N. *Let* ${}^t(1, \alpha, \beta)$ *be the column eigenvector of* N *for* λ *and we suppose that* λ *is cubic irrational and that* $1, \alpha, \beta$ *are linearly independent over* $\boldsymbol{Q}$. *Suppose furthermore* $(\alpha, \beta) \in X$.

If the convergents $\left(\frac{p_n}{q_n}, \frac{r_n}{q_n}\right)$ *of* (α, β) *satisfy condition* $(*)$, *then* (α, β) *is periodic.*

In view of Perron's result, this Theorem has the following corollary.

Corollary. *The convergents of* (α, β) *satisfy condition* $(*)$ *if and only if* (α, β) *is periodic and satisfies the condition* (1).

§2. Outline of the proof of the Theorem. The proof of the theorem is based on the following fundamental formula.

Fundamental formula. *Let us denote*

$$(\alpha_n, \beta_n) := T^n(\alpha, \beta).$$

Then the following formula holds:

$$(2.1)\quad {}^t(1, \alpha_n, \beta_n) = \frac{1}{\alpha\alpha_1\cdots\alpha_{n-1}} A(a_n, b_n)^{-1}\cdots A(a_1, b_1)^{-1}\,{}^t(1, \alpha, \beta).$$

Now put

$$Q_n := \begin{pmatrix} q_n & q_{n-2} & q_{n-1} \\ p_n & p_{n-2} & p_{n-1} \\ r_n & r_{n-2} & r_{n-1} \end{pmatrix}$$

where $(p_n/q_n, r_n/q_n)$ are convergents of (α, β) and

$$N_n := \begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & F_n \\ G_n & H_n & I_n \end{pmatrix} := Q_n^{-1} N Q_n,$$

and denote

$$P_n := q_n\alpha - p_n, \quad R_n := q_n\beta - r_n.$$

Under these notations, we find the exact forms of $A_n, B_n, \cdots, I_n$ as follows:

$$\begin{aligned} (2,\mathrm{A})\quad A_n &= (-q_{n-2}R_{n-1}+q_{n-1}R_{n-2})\alpha(bP_n+cR_n) \\ &+ (-q_{n-1}P_{n-2}+q_{n-2}P_{n-1})\beta(bP_n+cR_n) \\ &+ (-q_{n-1}R_{n-2}+q_{n-2}R_{n-1})(eP_n+fR_n) \\ &+ (-q_{n-2}P_{n-1}+q_{n-1}P_{n-2})(hP_n+iR_n) \end{aligned}$$

$$\begin{aligned} (2,\mathrm{B})\quad B_n &= (-q_{n-2}R_{n-1}+q_{n-1}R_{n-2})\alpha(bP_{n-2}+cR_{n-2}) \\ &+ (-q_{n-1}P_{n-2}+q_{n-2}P_{n-1})\beta(bP_{n-2}+cR_{n-2}) \\ &+ (-q_{n-1}R_{n-2}+q_{n-2}R_{n-1})(eP_{n-2}+fR_{n-2}) \\ &+ (-q_{n-2}P_{n-1}+q_{n-1}P_{n-2})(hP_{n-2}+iR_{n-2}) \end{aligned}$$

$$\begin{aligned} (2,\mathrm{C})\quad C_n &= (-q_{n-2}R_{n-1}+q_{n-1}R_{n-2})\alpha(bP_{n-1}+cR_{n-1}) \\ &+ (-q_{n-1}P_{n-2}+q_{n-2}P_{n-1})\beta(bP_{n-1}+cR_{n-1}) \\ &+ (-q_{n-1}R_{n-2}+q_{n-2}R_{n-1})(eP_{n-1}+fR_{n-1}) \\ &+ (-q_{n-2}P_{n-1}+q_{n-1}P_{n-2})(hP_{n-1}+iR_{n-1}) \end{aligned}$$

(2,D) $D_n = (-q_{n-1}R_n + q_nR_{n-1})\alpha(bP_n + cR_n)$
$+ (-q_nP_{n-1} + q_{n-1}P_n)\beta(bP_n + cR_n)$
$+ (-q_nR_{n-1} + q_{n-1}R_n)(eP_n + fR_n)$
$+ (-q_{n-1}P_n + q_nP_{n-1})(hP_n + iR_n)$

(2,E) $E_n = (-q_{n-1}R_n + q_nR_{n-1})\alpha(bP_{n-2} + cR_{n-2})$
$+ (-q_nP_{n-1} + q_{n-1}P_n)\beta(bP_{n-2} + cR_{n-2})$
$+ (-q_nR_{n-1} + q_{n-1}R_n)(eP_{n-2} + fR_{n-2})$
$+ (-q_{n-1}P_n + q_nP_{n-1})(hP_{n-2} + iR_{n-2})$

(2,F) $F_n = (-q_{n-1}R_n + q_nR_{n-1})\alpha(bP_{n-1} + cR_{n-1})$
$+ (-q_nP_{n-1} + q_{n-1}P_n)\beta(bP_{n-1} + cR_{n-1})$
$+ (-q_nR_{n-1} + q_{n-1}R_n)(eP_{n-1} + fR_{n-1})$
$+ (-q_{n-1}P_n + q_nP_{n-1})(hP_{n-1} + iR_{n-1})$

(2,G) $G_n = (-q_nR_{n-2} + q_{n-2}R_n)\alpha(bP_n + cR_n)$
$+ (-q_{n-2}P_n + q_nP_{n-2})\beta(bP_n + cR_n)$
$+ (-q_{n-2}R_n + q_nR_{n-2})(eP_n + fR_n)$
$+ (-q_nP_{n-2} + q_{n-2}P_n)(hP_n + iR_n)$

(2,H) $H_n = (-q_nR_{n-2} + q_{n-2}R_n)\alpha(bP_{n-2} + cR_{n-2})$
$+ (-q_{n-2}P_n + q_nP_{n-2})\beta(bP_{n-2} + cR_{n-2})$
$+ (-q_{n-2}R_n + q_nR_{n-2})(eP_{n-2} + fR_{n-2})$
$+ (-q_nP_{n-2} + q_{n-2}P_n)(hP_{n-2} + iR_{n-2})$

(2,I) $I_n = (-q_nR_{n-2} + q_{n-2}R_n)\alpha(bP_{n-1} + cR_{n-1})$
$+ (-q_{n-2}P_n + q_nP_{n-2})\beta(bP_{n-1} + cR_{n-1})$
$+ (-q_{n-2}R_n + q_nR_{n-2})(eP_{n-1} + fR_{n-1})$
$+ (-q_nP_{n-2} + q_{n-2}P_n)(hP_{n-1} + iR_{n-1})$.

We will prove the Theorem in showing the boundedness of $A_n, \cdots, I_n$.

Lemma 2.1. *A_n, C_n, D_n and G_n are bounded.*

Proof. From Fundamental formula, we see that

$$\alpha - \frac{p_n}{q_n} = \frac{(q_np_{n-2} - q_{n-2}p_n)\alpha_n + (q_np_{n-1} - q_{n-1}p_n)\beta_n}{(q_n + q_{n-2}\alpha_n + q_{n-1}\beta_n)q_n},$$

$$\beta - \frac{r_n}{q_n} = \frac{(q_np_{n-2} - q_{n-2}r_n)\alpha_n + (q_nr_{n-1} - q_{n-1}r_n)\beta_n}{(q_n + q_{n-2}\alpha_n + q_{n-1}\beta_n)q_n}.$$

and from the condition (∗) we see that for some K'

$$|P_n| < K'/\sqrt{q_n} \text{ and } |R_n| < K'/\sqrt{q_n} \text{ for all } n.$$

On the other hand, we see that

$$|q_nP_{n-1} - q_{n-1}P_n| = |q_np_{n-1} - q_{n-1}p_n| < K\sqrt{q_n},$$
$$|q_nR_{n-1} - q_{n-1}R_n| = |q_nr_{n-1} - q_{n-1}r_n| < K\sqrt{q_n} \text{ (by condition (∗))}.$$

Therefore from the formula (2,A), (2,C), (2,D) and (2,G), we see that A_n, C_n, D_n and G_n are bounded.

Lemma 2.2. *B_n, E_n and H_n are bounded if I_n and F_n are bounded.*

Proof. As N and $Q_n^{-1}NQ_n$ have the same eigenvalue we have following relations:

(1) $$A_n + E_n + I_n = a + e + i$$

(2) $$A_nE_n - B_nD_n + A_nI_n - C_nG_n + E_nI_n - F_nH_n = ae - bd + ai - cg + ei - fh$$

(3) $$A_n(E_nI_n - F_nH_n) + B_n(F_nG_n - D_nI_n) + C_n(D_nH_n - E_nG_n) = a(ei - fh) + b(fg - di) + c(dh - eg).$$

We denote the quantities $a+e+i$, $ae-bd+ai-cg+ei-fh$, and $a(ei-fh)+b(fg-di)+c(dh-eg)$ by k_1, k_2 and k_3 respectively. From (1), (2) and (3), we have the following formula:

$$(4)\quad \begin{pmatrix} B_n \\ H_n \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} C_nD_n - A_nF_n & F_n \\ D_nI_n - F_nG_n & -D_n \end{pmatrix} \times \begin{pmatrix} k_2 - k_1(A_n+I_n) + A_n^2 + A_nI_n + I_n^2 + C_nG_n \\ k_3 - (A_nI_n + C_nG_n)(k_1 - A_n - I_n) \end{pmatrix}$$

where

$$\Delta = \det\begin{pmatrix} -D_n & -F_n \\ -D_nI_n + F_nG_n & C_nD_n - A_nF_n \end{pmatrix}.$$

By using Lemma 2.1, the relations (1) and (4), and the corollary 2.4 given below we see that E_n, B_n and H_n are also bounded.

By the simple calculation we have the following lemma.

Lemma 2.3. *Let* $(1, \gamma, \delta)$ *be row eigenvectors of* N *for* λ. *Then* $\alpha, \beta, \gamma, \delta$ *satisfy the following equations*:

$$\begin{aligned}(2.2)\ & (-bce+b^2f+bci-c^2h)\alpha^3 \\ &+(-ace+ce^2-bcd+2abf-bfi+aci-bef-cei-c^2g+2chf)\alpha^2 \\ &+(-acd+2cde-bdf-cdi+a^2f-afi-aef+efi-hf^2-2cfg)\alpha \\ &+(cd^2-adf+dfi-gf^2)=0,\end{aligned}$$

$$\begin{aligned}(2.3)\ & -(-bce+b^2f+bci-c^2h)\beta^3 \\ &+(-abi+bi^2-bcg+2ach-che+abe-cih-bie-b^2d+2bhf)\beta^2 \\ &+(-abg+2bgi-cgh-bge+a^2h-ahe-aih+ihe-h^2f+2dhb)\beta \\ &+(bg^2-adh+ghe-dh^2)=0,\end{aligned}$$

$$\begin{aligned}(2.4)\ & (-edg+d^2h+dgi-g^2f)\gamma^3 \\ &+(-aeg+2adh+agi+e^2g-edh-egi-bdg-dih-g^2c+2hfg)\gamma^2 \\ &+(-abg+a^2h-aih+2beg-aeh+eih-bdh-bgi-fh^2+2chg)\gamma \\ &+(b^2g-abh+bih-ch^2)=0,\end{aligned}$$

$$\begin{aligned}(2.5)\ & -(-edg+b^2h+dgi-g^2f)\delta^3 \\ &+(-aid+2agf+ade+i^2d-igf-eid-cdg-gef-bd^2+2fhd)\delta^2 \\ &+(-acd+a^2f-aef+2cid-aif+eif-cgf-cde-hf^2+2bfd)\delta \\ &+(c^2d-acf+cef-bf^2)=0.\end{aligned}$$

Corollary 2.4. *The determinant* Δ *in Lemma 2.2 is not equal to zero.*

Proof. We find that the constant term of equation (2.2) is equal to $-\Delta$. From the assumption that $1, \alpha, \beta$ are linearly independent over $\boldsymbol{Q}$, α must be an irrational number in the cubic field $\boldsymbol{Q}(\lambda)$, i.e. a cubic irrational number. Therefore we have that $\Delta \neq 0$.

Lemma 2.5. *For the coefficients of the equations* (2.4) *and* (2.5) *we have the following equalities*:

$$\begin{aligned}(2.6)\ & (-edg+d^2h+dgi-g^2f) \\ &= d^3/\Delta\{((di-gf)/d)^3 - k_1((di-gf)/d)^2 + k_2((di-gf)/d) - k_3\},\end{aligned}$$

$$\begin{aligned}(2.7)\ & (c^2d-acf+cef-bf^2) \\ &= f^3/\Delta\{((af-cd)/f)^3 - k_1((af-cd)/f)^2 + k_2((af-cd)/f) - k_3\}.\end{aligned}$$

By the way we konw the following Lemma called Sturm's theorem.

Lemma 2.6 (Sturm's theorem on (0,1) for cubic polynomials). *For a given cubic polynomial* $f(x) = ax^3 + bx^2 + cx + d$, *the cardinality of real* *solution* $f(x)=0$ *in the interval* $(0,1)$ *is equal to* $V(0)-V(1)$, *where* $V(x_0)$ *is the signature of* $x=x_0$, *and the signature* $V(x)$ *is given from the sequence of polynomials*:

$$f_0(x)=ax^3+bx^2+cx+d \qquad f_1(x)=3x^2+2bx+c$$
$$f_2(x)=\frac{1}{9a}\{(2b^2-6ac)x+bc-3ad\} \quad f_3(x)=\frac{9a}{4(b^2-3ac)^2}D,$$

where D *is the discriminant of* $f(x)$.

Proof of the theorem. From Lemma 2.2 we will show that I_n and F_n are bounded. The proof is done by setting up $3^4+{}_4C_1(1+{}_4C_3+{}_4C_2\cdot{}_3C_2\cdot 2!+{}_4C_1\cdot 3!)=389$ cases with respect to the signatures of discriminant of λ and determinant Δ and the condition of $V(0)$, $V(1)$ of α, β, γ, δ. We only set up here the one case. Let us assume that

A-(1) the discriminant of λ is negative,

A-(2) all equations (2.2)-(2.5) for α, β, γ and δ satisfy the condition
$$V(0)=3 \text{ and } V(1)=2.$$

A-(3) the determinant Δ in Corollary 2.4 is positive.

On thse assumptions we will see that F_n and I_n are bounded. Indeed, from $V(0)=3$ we have $f_0(0)>0$, $f_1(0)<0$ and $f_2(0)>0$. From the fact that the equations (2.4) and (2.5) of γ and δ satisfies $V(0)=3$, and from the formulae (2.6) and (2.7) we see that

(2.8) $\quad d^3\{((di-gf)/d)^3-k_1((di-gf)/d)^2+k_2((di-gf)/d)-k_3\}>0,$

and

(2.9) $\quad f^3\{((af-cd)/f)^3-k_1((af-cd)/f)^2+k_2((af-cd)/f)-k_3\}>0.$

On the other hand, from the fact that the equations (2.2) and (2.3) of α and β satisfies $V(1)=2$ we also see that

(2.10) $\quad d^3\{((di-gf)/d)^3-k_1((di-gf)/d)^2+k_2((di-gf)/d)-k_3\}>K_1,$

and

(2.11) $\quad f^3\{((af-cd)/f)^3-k_1((af-cd)/f)^2+k_2((af-cd)/f)-k_3\}>K_2.$

From (2.8) and (2.10) we see that $(di-gf)/d$ is bounded. From (2.9) and (2.11) we see that $(af-cd)/f$ is bounded.

On the other hand, we know that A_n, C_n, D_n and G_n are bounded. From the fact that $\dfrac{A_nF_n-C_nD_n}{F_n}$ is bounded, we know that F_n is bounded. From the fact that $\dfrac{D_nI_n-G_nF_n}{D_n}$ is bounded, we know that I_n is also bounded.

By similar discussions for the other 389-1 cases, we have the boundedness of these integer coefficients. Therefore, there exist $m_0>n_0$ such that
$$N_{m_0}=N_{n_0}.$$

On the other hand, we know from Fundamental formula that
$$N_n^t(1,\alpha_n,\beta_n)=Q_n^{-1}NQ_n^t(1,\alpha_n,\beta_n)=1/(\alpha\alpha_1\cdots,\alpha_{n-1})Q_n^{-1}N^t(1,\alpha,\beta)$$
$$=\lambda/(\alpha\alpha_1\cdots,\alpha_{n-1})Q_n^{-1}{}^t(1,\alpha,\beta)=\lambda^t(1,\alpha_n,\beta_n).$$

Therefore we see that ${}^t(1,\alpha_{n_0},\beta_{n_0})$ and ${}^t(1,\alpha_{m_0},\beta_{m_0})$ are eigen vectors of λ for $N_{m_0}=N_{n_0}$ and so we have $(\alpha_{n_0},\beta_{n_0})=(\alpha_{m_0},\beta_{m_0})$.

References

[1] O. Perron: Grundlagen für eine Theorie des Jacobischen Kettenbruch-algorithms. Math. Ann., **64**, 1–76 (1907).

[2] F. Schweiger: The metrical theory of Jacobi-Perron algorithm. Lect. Notes in Math., vol. 334, Springer (1973).