73. On the Rational Approximations to tanh $\frac{1}{k}$. II

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§1. Introduction. Throughout this note, we assume that k, p and q are integers with $k \geq 1$ and $q \geq 2$, and assume that p_n/q_n is the n-th convergent of tanh $\frac{1}{k}$. In the previous paper [4], we proved the following theorem.

Theorem A. *For any* $\varepsilon > 0$, *there is an infinity of solutions of the inequality*

$$(1) \qquad |\tanh\frac{1}{k} - \frac{p}{q}| < \left(\frac{1}{2k} + \varepsilon\right)\frac{\log\log q}{q^2 \log q}$$

in integers p *and* q. *Further, there exists a number* $q' = q'(k, \varepsilon)$ *such that*

$$|\tanh\frac{1}{k} - \frac{p}{q}| > \left(\frac{1}{2k} - \varepsilon\right)\frac{\log\log q}{q^2 \log q}$$

for all integers p *and* q *with* $q \geq q'$.

The second statement of the theorem shows that the constant $\frac{1}{2k}$ in the inequality (1) is 'best possible' in the sense that it can not be replaced by any smaller number. Nonetheless, the inequality (1) may be improved, in that $\frac{1}{2k} + \varepsilon$ may be replaced by $\frac{1}{2k}$, and it is the purpose of this note to establish this result, thus giving the

Theorem. *There is an infinity of solutions of the inequality*

$$|\tanh\frac{1}{k} - \frac{p}{q}| < \frac{1}{2k}\frac{\log\log q}{q^2 \log q}$$

in integers p *and* q.

§2. The proof of Theorem. The continued fraction of tanh $\frac{1}{k}$ is

$$\tanh\frac{1}{k} = [a_0, a_1, a_2, a_3, \cdots] = [0, k, 3k, 5k, \cdots].$$

In other words, $a_0 = 0$ and $a_n = k(2n-1)$ for $n \geq 1$. Since $q_{n+1} = a_{n+1}q_n + q_{n-1} > k(2n+1)q_n$, we have

$$|\tanh\frac{1}{k} - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}} < \frac{1}{k(2n+1)q_n^2}.$$

Now

$$\log q_n = n\log n + O(n) = n\log n\,\{1 + O(1/(\log n))\},$$

so

$$\begin{aligned}\log\log q_n &= \log n + \log\log n + O(1/(\log n)) \\ &= \frac{\log q_n}{n} + \log\log n + O(1),\end{aligned}$$

and hence

$$\frac{1}{n} < \frac{\log\log q_n}{\log q_n}$$

for all sufficiently large n. Thus

$$\left| \tanh \frac{1}{k} - \frac{p_n}{q_n} \right| < \frac{1}{2knq_n^2} < \frac{\log\log q_n}{2kq_n^2 \log q_n}$$

for an infinity of p_n and q_n, as asserted.

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