72. *Triangles and Elliptic Curves. III*

By Takashi ONO

Department of Mathematics, The Johns Hopkins University, U. S. A.

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This is a continuation of my preceding papers [2], [3] which will be referred to as (I), (II) in this paper. In (II), to each triple (l, m, n) of independent linear forms on $\bar{k}^3$, $\bar{k}$ being the algebraic closure of a field k of characteristic not 2, we associated a space $T = \{t \in \bar{k}^3 ; (l^2 - m^2)(m^2 - n^2)(n^2 - l^2) \neq 0\}$ and studied a relationship of T to a family of plane elliptic curves. In this paper, we shall obtain a parametrization of T by classical elliptic functions when $k = \boldsymbol{C}$.[1)]

§1. Still over $\bar{k}$. Let $\Omega = \{\omega = (M, N) \in \bar{k} \times \bar{k} ; MN(M - N) \neq 0\}$. For each $\omega \in \Omega$, let

(1.1) $$E_0(\omega) = \{t \in \bar{k}^3 ; n^2 + M = l^2,\ n^2 + N = m^2\},$$

an affine part of an elliptic curve in $P^3(\bar{k})$. Then we obtain a surjective map $p : T \to \Omega$ given by

(1.2) $$p(t) = (l^2 - n^2,\ m^2 - n^2).$$

Since we observe that

(1.3) $$p^{-1}(\omega) = E_0(\omega), \quad \omega \in \Omega,$$

we have

(1.4) $$T = \bigcup_{\omega \in \Omega} E_0(\omega) \text{ (disjoint)}.$$

To each $\omega = (M, N)$ we associate an elliptic curve E_ω in $\boldsymbol{P}^2(\bar{k})$ given (affinely) by

(1.5) $$E_\omega : y^2 = x(x + M)(x + N).$$

Then we observe that a map $\pi_0 : E_0(\omega) \to E_\omega$, $\omega = (M, N) \in \Omega$, defined by

(1.6) $$\pi_0(t) = (n^2,\ lmn)$$

makes sense, for $x(x + M)(x + N) = n^2(n^2 + M)(n^2 + N) = (lmn)^2 = y^2$.

§2. The map Θ_τ. Denote by $\vartheta_i(v \mid \tau)$, $i = 0, 1, 2, 3$, $v \in \boldsymbol{C}$, $\tau \in \mathcal{H}$, the upper half plane, the Jacobi theta functions. When τ is fixed, we write $\vartheta_i(v)$ instead of $\vartheta_i(v \mid \tau)$. We write $\vartheta_i = \vartheta_i(0) = \vartheta_i(0 \mid \tau)$ for simplicity. The lattice $L_\tau = \boldsymbol{Z} + \boldsymbol{Z}\tau$ is the set of zeros of $\vartheta_1(v)$ and $\vartheta_i(v)$ and $\vartheta_j(v)$ have no common zeros if $i \neq j$. We introduce the following notation:

$$k = k(\tau) = \left(\frac{\vartheta_2}{\vartheta_3}\right)^2,\ k' = k'(\tau) = \left(\frac{\vartheta_0}{\vartheta_3}\right)^2,\ \sqrt{k} = \frac{\vartheta_2}{\vartheta_3},\ \sqrt{k'} = \frac{\vartheta_0}{\vartheta_3},$$

$$\sqrt{\frac{k'}{k}} = \frac{\sqrt{k'}}{\sqrt{k}} = \frac{\vartheta_0}{\vartheta_2},\ K = K(\tau) = \frac{\pi}{2}\vartheta_3^2, \quad u = 2Kv = 2K(\tau)v,$$

where u is taken to be a new complex variable.

Now define a map $\Theta_\tau : \boldsymbol{C} \to P^3(\boldsymbol{C})$ by

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1) See [1] and/or [5] for standard notations.

(2.1) $$\Theta_\tau(u) = \left(\vartheta_0(v) : \frac{1}{\sqrt{k}}\vartheta_1(v) : \sqrt{\frac{k'}{k}}\vartheta_2(v) : \sqrt{k'}\,\vartheta_3(v)\right).$$

Then Θ_τ induces an analytic group isomorphism:

(2.2) $$\boldsymbol{C}/(4K(\tau)L_\tau) \approx E(-1, -k^2(\tau)),$$

where $E(M, N)$ denotes the space elliptic curve defined by

(2.3) $$E(M, N) = \{x = (x_0 : x_1 : x_2 : x_3) \in P^3(\boldsymbol{C})\,;\, x_0^2 + Mx_1^2 = x_2^2,\ x_0^2 + Nx_1^2 = x_3^2\},$$

where $M, N \in \boldsymbol{C}$ with $MN(M-N) \neq 0$ ([4] Theorem 4.2).

Next, we need Jacobi's elliptic functions, fixing a $\tau \in \mathscr{H}$:

$$sn(u, k) = \frac{1}{\sqrt{k}}\frac{\vartheta_1(v)}{\vartheta_0(v)},\ cn(u, k) = \sqrt{\frac{k'}{k}}\frac{\vartheta_2(v)}{\vartheta_0(v)},\ dn(u, k) = \sqrt{k'}\frac{\vartheta_3(v)}{\vartheta_0(v)},$$

with relations

(2.4) $$cn^2(u, k) = 1 - sn^2(u, k),\ dn^2(u, k) = 1 - k^2 sn^2(u, k).$$

Since $sn(u, k)$ does not vanish on $\boldsymbol{C} - (4k)L_\tau$, the following map $\Theta_\tau^* : \boldsymbol{C} - (4k)L_\tau \to \boldsymbol{C}^3$ given by

(2.5) $$\Theta_\tau^*(u) = \left(\frac{1}{sn(u, k)}, \frac{cn(u, k)}{sn(u, k)}, \frac{dn(u, k)}{sn(u, k)}\right),$$

makes sense. Finally, we call ι an embedding of $\boldsymbol{C}^3$ into $P^3(\boldsymbol{C})$ given by

(2.6) $$(x, y, z) \mapsto (x : 1 : y : z)$$

We verify the commutativity of the following diagram easily:

(2.7) $$\begin{array}{ccc} \boldsymbol{C} & \xrightarrow{\Theta_\tau} & P^3(\boldsymbol{C}) \\ \cup & & \cup\,\iota \\ \boldsymbol{C} - (4K)L_\tau & \xrightarrow{\Theta_\tau^*} & \boldsymbol{C}^3 \end{array}$$

§3. A covering $S \to T$. Returning to the space T in the beginning of the paper, with $k = \boldsymbol{C}$ this time, denote by Φ the matrix in $GL_3(\boldsymbol{C})$ determined by the condition

(3.1) $$\Phi t = \begin{pmatrix} l(t) \\ m(t) \\ n(t) \end{pmatrix},\quad t = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \boldsymbol{C}^3.$$

Therefore T is determined by Φ. From now on, we denote by T_1 the space T corresponding to $\Phi = 1 \in GL_3(\boldsymbol{C})$. Note that $T_1 = \Phi T$.[2)] In order to make a covering space S of T as small as possible, we first let

(3.2) $$\boldsymbol{C}^\# = \{\alpha = re^{i\theta}\,;\, r > 0,\ 0 \leqq \theta < \pi\}.$$

Next let

(3.3) $$D(2) = D_1 \cup D_2$$

where $D_1 = \{z \in \mathscr{H}\,; 0 < \mathrm{Re}\, z \leqq 1,\ |z - 1/2| \geqq 1/2\}$, $D_2 = \{z \in \mathscr{H}\,; -1 < \mathrm{Re}\, z \leqq 0,\ |z + 1/2| > 1/2\}$. In other words, $D(2)$ is the standard fundamental domain for $\Gamma(2)\backslash\mathscr{H}$, with

2) While working over algebraically closed fields such as $\boldsymbol{C}$, we may assume that $\Phi = 1$ without loss of generality. However, the choice of $\Phi \neq 1$ matters to us when other fields are considered. See, e.g., (I) and (1.7) of (II) where $\Phi = \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ appears in connection with euclidean geometry.

$$\Gamma(2) = \{A \in SL_2(\boldsymbol{Z})\ ; A \equiv 1 \bmod 2\}.$$

Finally, we let

(3.4) $\quad S = \{s = (\alpha, u, \tau) \in \boldsymbol{C}^{\#} \times \boldsymbol{C} \times D(2)\ ; u \notin 4k(\tau)L_\tau\}.$

Defining a map $\psi : S \to T$ amounts to defining a map $\psi_1 : S \to T_1$ such that $\psi_1 = \Phi\psi$. So let us consider a map $\psi_1 : S \to \boldsymbol{C}^3$ given by

$$\psi_1(S) = \alpha \begin{pmatrix} \dfrac{cn(u, k(\tau))}{sn(u, k(\tau))} \\ \dfrac{dn(u, k(\tau))}{sn(u, k(\tau))} \\ \dfrac{1}{sn(u, k(\tau))} \end{pmatrix} \tag{3.5}$$

We shall show that (3.5) is a covering $\psi_1 : S \to T_1$ we are looking for. To be more precise, we prove the following three statements (3.6)-(3.8).

(3.6) $\quad \psi_1(S) \subset T_1.$

Proof. Writing $\psi_1(s) = t = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, we have to show that $(a^2 - b^2)(b^2 - c^2)(c^2 - a^2) \neq 0$. This follows from (2.4), (3.5) and the property $k^2(\tau) \neq 0,1$. **Q.E.D.**

(3.7) $\quad \psi_1 : S \to T_1$ *is surjective.*

Proof. Take any $t = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in T_1$. By (1.4), there is an $\omega = (M, N) \in \Omega$ such that $t \in E_0(\omega)$, i.e., $c^2 + M = a^2$, $c^2 + N = b^2$. Put $\alpha = \sqrt{-M}$. Since $N/M \neq 0,1$ and k^2 is *the* modular function for $\Gamma(2)$ we can find a (unique) $\tau \in D(2)$ such that $k^2(\tau) = N/M$. By the above choice of α we have $\left(\frac{c}{\alpha}\right)^2 - 1 = \left(\frac{a}{\alpha}\right)^2$, $\left(\frac{c}{\alpha}\right)^2 - k^2(\tau) = \left(\frac{b}{\alpha}\right)^2$ which means, by (2.3), that $\left(\frac{c}{\alpha} : 1 : \frac{a}{\alpha} : \frac{b}{\alpha}\right) \in E(-1, -k^2(\tau))$ and so, by (2.2), there is a $u \in \boldsymbol{C} - 4K(\tau)L_\tau$ such that $\Theta_\tau(u) = \left(\frac{c}{\alpha} : 1 : \frac{a}{\alpha} : \frac{b}{\alpha}\right)$. Now set $S = (\alpha, u, \tau)$. Then, from (2.5)-(2.7), (3.5), it follows that $\left(\frac{1}{sn(u, k(\tau))}, \frac{cn(u, k(\tau))}{sn(u, k(\tau))}, \frac{dn(u, k(\tau))}{sn(u, k(\tau))}\right) = \Theta_\tau^*(u) = \left(\frac{c}{\alpha}, \frac{a}{\alpha}, \frac{b}{\alpha}\right)$, i.e., $\psi_1(s) = t$. **Q.E.D.**

(3.8) $\quad$ *For* $s_i = (\alpha_i, u_i, \tau_i) \in S$, $i = 1,2$, $\psi_1(s_1) = \psi_1(s_2)$ *if and only if* $\alpha_1 = \alpha_2$, $\tau_1 = \tau_2$ *and* $u_1 \equiv u_2 \bmod 4K(\tau_1)L_{\tau_1}$.

Proof. The if-part is obvious as $4K(\tau_1)L_{\tau_1}$ is the period lattice for $sn(u, k(\tau_1))$, etc. Conversely, suppose that $\psi_1(s_1) = \psi_1(s_2)$. Comparing squares of components of this vector equation, we find, using (2.4), that $\alpha_2^2 = \alpha_1^2$ and $k^2(\tau_1) = k^2(\tau_2)$. Hence we have $\alpha_2 = \alpha_1$ and $\tau_2 = \tau_1$ because $\alpha_i \in \boldsymbol{C}^{\#}$ and $\tau_i \in D(2)$. Therefore, putting $\tau = \tau_1 = \tau_2$, $\psi_1(s_1) = \psi_1(s_2)$ implies $\Theta_\tau^*(u_1) = \Theta_\tau^*(u_2)$ and so $\Theta_\tau(u_1) = \Theta_\tau(u_2)$ by (2.7). Therefore we obtain $u_1 \equiv u_2 \bmod 4K(\tau)L_\tau$ by (2.2). **Q.E.D.**

Remark. For each $\tau \in D(2)$ we write $P_\tau^* = P_\tau - \{0\}$ where P_τ is the standard fundamental domain for $C/(4K(\tau)L_\tau)$. Then the statements (3.6)-(3.8) means that for the space T (determined by Φ) the map $\psi(=\Phi^{-1}\psi_1)$ induces a bijection

$$T \approx C^{\#} \times \bigcup_{\tau \in D(2)} P_\tau^*, \tag{3.9}$$

an analytic parametrization of the complement of six lines $(l^2 - m^2)(m^2 - n^2)(n^2 - l^2) = 0$ in C^3.

§4. Differentiation. We shall look at analytically the map π_0 in (1.6). Let T be given by Φ as in (3.1). If $\psi(s) = t$, $s \in S$, $t \in T$, then $\psi_1(s) = \Phi t$. By (3.5), we obtain a system of equations:

$$l(t) = \alpha \frac{cn(u, k(\tau))}{sn(u, k(\tau))},\ m(t) = \alpha \frac{dn(u, k(\tau))}{sn(u, k(\tau))},\ n(t) = \alpha \frac{1}{sn(u, k(\tau))}, \tag{4.1}$$

If we let $x = x(s) = n^2(t)$, $y = y(s) = l(t)m(t)n(t)$, then there is a relation

$$y^2 = x(x + M)(x + N) \tag{4.2}$$

with $M = l^2(t) - n^2(t)$, $N = m^2(t) - n^2(t)$. Substituting (4.1) in (4.2), we obtain, by (2.4)

$$M = -\alpha^2,\ N = -k^2(\tau)\alpha^2, \tag{4.3}$$

i.e., M, N do not involve u. Hence, for fixed α, τ, (4.2) is a plane elliptic curve. We see easily that

$$y = \frac{\alpha}{2}\frac{\partial x}{\partial u}. \tag{4.4}$$

Substituting (4.4) in (4.2), we obtain

$$\alpha^2\left(\frac{\partial x}{\partial u}\right)^2 = 4x(x - \alpha^2)(x - k^2\alpha^2). \tag{4.5}$$

References

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