9. *A Serre Type Theorem for Affine Lie Superalgebras and Their Quantized Enveloping Superalgebras*

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Introduction. In 1985, Drinfeld [1] and Jimbo [2] introduced an h-adic topological $C[[h]]$-Hopf algebra $U_h\mathfrak{G}$ associated to a Kac-Moody Lie algebra $\mathfrak{G}$, which is now known as the *quantum group* or the *quantized enveloping algebra*. The algebra $U_h\mathfrak{G}$ is defined by generators and binary relations called the *q-Serre relations*. Let $U_h\mathfrak{B}_+$ and $U_h\mathfrak{B}_-$ be the positive and negative Borel-type Hopf subalgebras of $U_h\mathfrak{G}$. Drinfeld showed that there is a non-degenerate bilinear form $\langle,\rangle : U_h\mathfrak{B}_+ \otimes U_h\mathfrak{B}_- \to C((h))$ such that, under $\langle,\rangle$, $U_h\mathfrak{B}_-$ can be identified with a dual Hopf algebra of $U_h\mathfrak{B}_+$ with the opposite comultiplication. (Speaking more strictly, we must shift the topologies of $U_h\mathfrak{B}_+$ and $U_h\mathfrak{B}_-$ and replace $U_h\mathfrak{B}_+$ and $U_h\mathfrak{B}_-$ with certain Hopf subalgebras. See [8] and §3 of this note.) He also proved the existence of the *universal R-matrix* of $U_h\mathfrak{G}$ by using $\langle,\rangle$. His method is called the *guantum double construction.*

The purposes of this note are to exhibit the following results:

(i) *A Serre type theorem for an affine Lie superalgebra* $\mathfrak{g}$. We give defining relations of $\mathfrak{g}$ satisfied by the Chevalley generators. We need not only binary relations but also trinormial and quadrinomial relations.

(ii) *A definition of the quantized enveloping superalgebra* $U_h\mathfrak{g}$ *associated to* $\mathfrak{g}$. We define the h-adic topological $C[[h]]$-Hopf superalgebra $U_h\mathfrak{g}$ by using generators and relations.

(iii) *The existence of the universal R-matrix of a Hopf algebraization* $U_h\mathfrak{g}^\sigma$ *of* $U_h\mathfrak{g}$. We show this fact by using the quantum double construction for $U_h\mathfrak{g}^\sigma$. (In [8], we gave an embedding $(\cdot)^\sigma$ from the category of Hopf superalgebras to the category of Hopf algebras. This fact might be known to experts.)

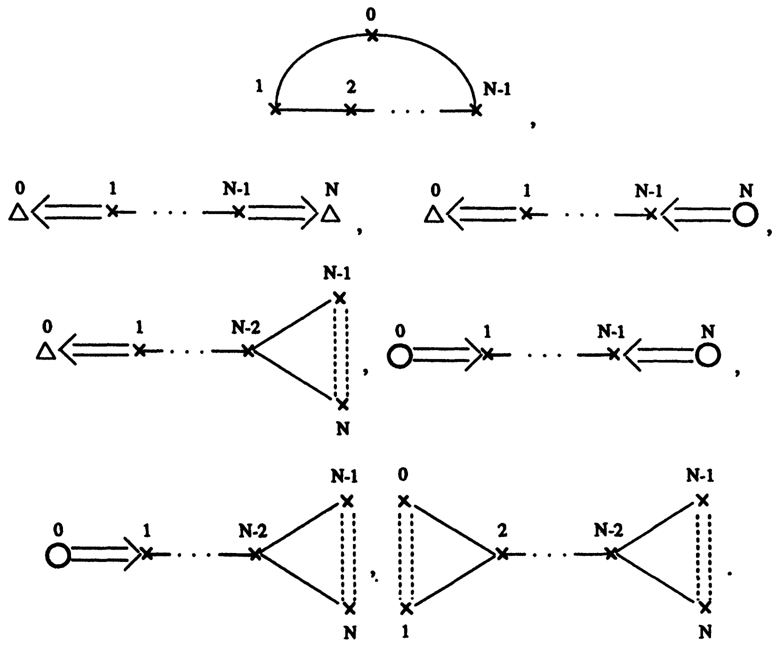
(iv) A topological freeness of the $C[[h]]$-module $U_h\mathfrak{g}$.

We have already shown the corresponding results for the finite dimensional simple Lie superalgebras of type A-G in [7], [8], [9].

Let $\mathfrak{g} = \mathfrak{g}(\mathcal{E}, \Pi, p)$ be the Kac-Moody Lie superalgebra defined with the datum $(\mathcal{E}, \Pi, p)$, the dual space $\mathcal{E}$ of a Cartan subalgebra $\mathfrak{h}$, the set $\Pi \subset \mathcal{E}$ of simple roots $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ and the parity function $\rho : \Pi \to \{0,1\}$. We first define $\mathfrak{g}$ abstractly by imitating the definition of the Kac-Moody Lie algebra given in §1.3 of Kac's text book [3]. Unfortunately, in the case of Lie superalgebras, the terminology "affine type" seems not to have been given the definite meaning, yet. For the present, we say that $\mathfrak{g}(\mathcal{E}, \Pi, p)$ is of affine type if the Dynkin diagram $\Gamma = \Gamma(\mathcal{E}, \Pi, p)$ can be found in our reference [4]. In this note, we further assume that $n \geq 4$ if Γ is not of type $G_3^{(1)}$ nor $D(2,1\,;x)^{(1)}$ and that Γ is of distinguished type if Γ is of type $F_4^{(1)}$, $G_3^{(1)}$ and $D(2,1\,;x)^{(1)}$. In fact, the Hopf superalgebra structure of $U_h\mathfrak{g}(\mathscr{E}, \Pi, p)$ seems to depend on the choice of the datum $(\mathscr{E}, \Pi, p)$. For the terminologies in this note, see [7], [8] and their references. Details omitted here will be published elsewhere.

1. Dynkin diagrams of affine Lie superalgebras. Let $\mathscr{E} = (\bigoplus_{i=1}^{N} \boldsymbol{C}\varepsilon_i) \oplus \boldsymbol{C}\delta \oplus \boldsymbol{C}\Lambda_0$ be the $N+2$ dimensional $\boldsymbol{C}$-vector space. Define the symmetric bilinear form $(,)\,;\mathscr{E} \times \mathscr{E} \to \boldsymbol{C}$ by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(\delta, \Lambda_0) = 1$ and $(\varepsilon_i, \delta) = (\varepsilon_i, \Lambda_0) = (\delta, \delta) = (\Lambda_0, \Lambda_0) = 0$. Let $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ be a linearly independent subset of $\mathscr{E}$, and $p : \Pi \to \boldsymbol{Z}/2\boldsymbol{Z}$ a function. We call α_i the simple roots and p the parity function. In this note, we consider the datum $(\mathscr{E}, \Pi, p)$ associated to one of the following Dynkin diagrams $\Gamma = \Gamma(\mathscr{E}, \Pi, p)$ listed below as (i) and (ii). In Γ, the i-th dot is ○, ⊗ or ● if and only if $((\alpha_i, \alpha_i), p(\alpha_i)) \in \boldsymbol{C}^{\times} \times \{0\}$, $\{(0,1)\}$ or $\boldsymbol{C}^{\times} \times \{1\}$ respectively where $\boldsymbol{C}^{\times} = \boldsymbol{C} \setminus \{0\}$. The i-th dot is joined to the j-th dot $(i \neq j)$ by $|(\alpha_i, \alpha_j)|$ edges. Moreover the arrow points to the smaller of $|(\alpha_i, \alpha_i)|$ and $|(\alpha_j, \alpha_j)|$. In this note, we assume that *the number of the dots of Γ is more than or equals to* 5 *if Γ is not of type $G_3^{(1)}$ or $D(2,1\,;x)^{(1)}$.*

(i) *ABCD types.* In this stage, the simple roots α_i satisfy that $\alpha_0 \in \{\delta - \bar{\varepsilon}_1 + \bar{\varepsilon}_N, \delta - \bar{\varepsilon}_1 - \bar{\varepsilon}_2, \delta - \bar{\varepsilon}_1, \delta - 2\bar{\varepsilon}_1\}$, $\alpha_i = \bar{\varepsilon}_i - \bar{\varepsilon}_{i+1}$ $(1 \leq i \leq N-1)$, and $\alpha_N \in \{\bar{\varepsilon}_N, 2\bar{\varepsilon}_N, \bar{\varepsilon}_{N-1} + \bar{\varepsilon}_N\}$. Here $\bar{\varepsilon}_i = \sqrt{\pm 1}\,\varepsilon_i$ where the sign can be choosen arbitrarily. The dot × stands for ○ or ⊗. The dot △ stands for ○ or ●. The edge × ┄┄ × stands for ⊗═⊗ or ○ ○.



(ii) *Exceptional types.* Here the element written below the i-th dot denotes the i-th simple root α_i.

$F_4^{(1)}$

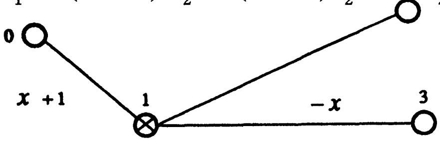
$\varepsilon_2-\varepsilon_3 \quad \varepsilon_3-\varepsilon_4 \quad \varepsilon_4 \quad \frac{1}{2}(\sqrt{-3}\,\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4) \quad \delta-\sqrt{-3}\,\varepsilon_1,$

$G_3^{(1)}$

$\delta-2\sqrt{-2}\varepsilon_4 \quad \varepsilon_2-\varepsilon_3+\sqrt{-2}\,\varepsilon_4 \quad \varepsilon_1-\varepsilon_2 \quad -2\varepsilon_1+\varepsilon_2+\varepsilon_3,$

$D(2,1;x)^{(1)}$ $(x\in \boldsymbol{C},\ x\neq 0,\ -1)$

$\delta-\sqrt{-1}(x-c+2)\,\varepsilon_1-(c-1)\,\varepsilon_2-(c-1)\varepsilon_2$ $\quad \varepsilon_2-\varepsilon_3$



$\sqrt{-1}\,\varepsilon_1-\varepsilon_2 \qquad \sqrt{-1}\,(x-c)\,\varepsilon_1+c\varepsilon_2+c\varepsilon_3$

where $c=-x+\sqrt{2x(x+1)}$.

2. A Serre type theorem for $\mathfrak{g}(\mathscr{E},\Pi,p)$ and defining relations of $U_h\mathfrak{g}(\mathscr{E},\Pi,p)$. Put $\mathfrak{h}=\mathscr{E}^*$. We call $\mathfrak{h}$ the Cartan subalgebra. Let $H_\nu\in\mathfrak{h}$ be the element such that $\mu(H_\nu)=(\mu,\nu)$ for any $\mu\in\mathscr{E}$. Let $\boldsymbol{C}[[h]]$ be the $\boldsymbol{C}$-algebra of formal power series in the indeterminate h.

Definition. Let $(\mathscr{E},\Pi,p)$ be the datum in Section 1. We define the h-adic topological $\boldsymbol{C}[[h]]$-superalgebra $U_h\mathfrak{g}(\mathscr{E},\Pi,p)$ by generators $H\in\mathfrak{h}$, E_i, $F_i(0\leq i\leq n)$ with parities $p(H)=0, p(E_i)=p(F_i)=p(\alpha_i)$ and relations:

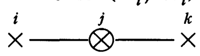
(2.1) $$[H,H']=0 \text{ if } H,H'\in\mathfrak{h},$$

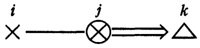
(2.2) $$[H,E_i]=\alpha_i(H)E_i,\ [H,F_i]=-\alpha_i(H)F_i,$$

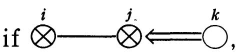
(2.3) $$[E_i,F_j]=\delta_{ij}\,\mathrm{sh}(hH_{\alpha_i})/\mathrm{sh}(h),$$

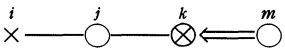
(2.4) (i) $[E_i,E_j]=0$ if $(\alpha_i,\alpha_j)=0$,

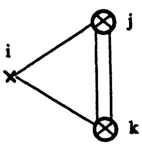
(ii) $(ad_h(E_i))^{1-a_{ij}}(E_j)=0$ if $(\alpha_i,\alpha_i)\neq 0$,

(iii) $[E_{ijk},E_j]=0$ if  with $(\alpha_i,\alpha_j)+(\alpha_j,\alpha_k)=0$

or ,

(iv) $[[E_{ij},E_{ijk}]_h,E_j]=0$ if ,

(v) $[[E_{ijkmkj},E_k]=0$ if ,

(vi) $E_{ijk}=E_{ikj}$ if 

,

(vii) $E_{043243} = [2]E_{043234}$ for type $F_4^{(1)}$,
(viii) $[2]E_{012312} = [3]E_{012321}$ for type $G_3^{(1)}$,
(ix) $[[E_{10}, E_{12}]_h, E_{13}] = [x][[E_{10}, E_{13}]_h, E_{12}]$ for type $D(2,1;x)^{(1)}$,
(2.5) the relations (2.4) with E_i's replaced by F_i's,
where we used the following notations: For $x \in \boldsymbol{C}$, put $[x] = (e^{xh} - e^{-xh})/(e^h - e^{-h}) \in \boldsymbol{C}[[h]]$. Put $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$. Put $Q^+ = \oplus \boldsymbol{Z}_{\geq 0}\alpha_i$. For $\lambda, \mu \in Q^+$, set $[X_\lambda, Y_\mu]_h = X_\lambda Y_\mu - (-1)^{p(\lambda)p(\mu)}e^{-(\lambda,\mu)h}Y_\mu X_\lambda$ where X_λ and X_μ satisfy $[H_\nu, X_\lambda] = (\nu, \lambda)X_\lambda$ and $[H_\nu, X_\mu] = (\nu, \mu)X_\mu$. Define $ad_h(X_\lambda)(X_\mu) = [X_\lambda, X_\mu]_h$ and $(ad_h(X_\lambda))^m(X_\mu) = ad_h(X_\lambda)(ad_h(X_\lambda))^{m-1}(X_\mu))$. Put $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \in \boldsymbol{Z}_{\leq 0}$ if $(\alpha_i, \alpha_i) \neq 0$. Let $E_{ijk\cdots} = [\cdots[[E_i, E_j]_h, E_k]_h, \cdots]_h$.

We are going to define the Kac-Moody Lie superalgebra $\mathfrak{g}(\mathscr{E}, \Pi, p)$ by imitating the definition of the Kac-Moody Lie algebra in §1.3 of Kac's text book [3]: Let $\tilde{\mathfrak{g}}(\mathscr{E}, \Pi, p)$ be the $\boldsymbol{C}$-Lie superalgebra defined by generators $H \in \mathfrak{h}$, E_i, F_i with parities $p(H) = 0$, $p(E_i) = p(F_i) = p(\alpha_i)$ and the relations obtained by substituting 0 for h in (2.1-3). We define $\mathfrak{g}(\mathscr{E}, \Pi, p)$ as the quotient Lie superalgebra $\tilde{\mathfrak{g}}(\mathscr{E}, \Pi, p)/\mathfrak{r}$ where $\mathfrak{r}$ is the ideal maximal among the ones such that $\mathfrak{r} \cap \mathfrak{h} = 0$.

Theorem A. *Let $(\mathscr{E}, \Pi, p)$ be the datum in Section* 1. *Defining relations of $\mathfrak{g}(\boldsymbol{E}, \Pi, p)$ satisfied by the generators $H \in \mathfrak{h}$, E_i, F_i $(0 \leq i \leq n)$ are given by substituting* 0 *for h in* (2.1-5).

3. Existence of a universal $\boldsymbol{R}$-matrix associated to $U_h\mathfrak{g}(\mathscr{E}, \Pi, p)$. In Theorem 2.9.4 in [8], we defined the h-adic topological $\boldsymbol{C}[[h]]$-Hopf superalgebra $U_h(\mathscr{E}, \Pi, p)$ in an abstract manner. By showing $U_h\mathfrak{g}(\mathscr{E}, \Pi, p) \cong U_h(\mathscr{E}, \Pi, p)$, we have Theorems B and C.

Theorem B. (i) *As a $\boldsymbol{C}[[h]]$-module. $U_h\mathfrak{g}(\boldsymbol{E}, \Pi, p)$ is topologically free.*

(ii) *$U_h\mathfrak{g}(\mathscr{E}, \Pi, p)$ is a topological Hopf superalgebra with coproduct Δ such that $\Delta(H) = H \otimes 1 + 1 \otimes H$, $\Delta(E_i) = E_i \otimes 1 + \exp(hH_{\alpha_i}) \otimes E_i$, $\Delta(F_i) = F_i \otimes \exp(-hH_{\alpha_i}) + 1 \otimes F_i$,*

(iii) *$U_h\mathfrak{g}(\mathscr{E}, \Pi, p)/hU_h\mathfrak{g}(\mathscr{E}, \Pi, p) \cong U\mathfrak{g}(\mathscr{E}, \Pi, p)$ as $\boldsymbol{C}$-Hopf superalgebras where $U\mathfrak{g}(\mathscr{E}, \Pi, p)$ is the universal enveloping algebra of $\mathfrak{g}(\mathscr{E}, \Pi, p)$.*

Let σ be the generator of $\boldsymbol{Z}/2\boldsymbol{Z}$. Let $\boldsymbol{Z}/2\boldsymbol{Z}$ act on $U_h\mathfrak{g}(\mathscr{E}, \Pi, p)$ by $\sigma.X = (-1)^{p(X)}X$ $(X \in U_h\mathfrak{g}(\mathscr{E}, \Pi, p))$. We define the algebra $U_h\mathfrak{g}(\mathscr{E}, \Pi, p)^\sigma$ as the crossed product $U_h\mathfrak{g}(\mathscr{E}, \Pi, p) \otimes_\sigma \boldsymbol{Z}/2\boldsymbol{Z}$. We denote the element $X \otimes \sigma^c \in U_h\mathfrak{g}(\mathscr{E}, \Pi, p) \otimes_\sigma \boldsymbol{Z}/2\boldsymbol{Z}$ by $X\sigma^c$. Then we can regard $U_h\mathfrak{g}(\mathscr{E}, \Pi, p)^\sigma$ as the Hopf algebra with the coproduct Δ^σ such that $\Delta^\sigma(X\sigma^c) = \sum_i X_i^{(1)}\sigma^{(p(X_i^{(2)})+c)} \otimes X_i^{(2)}\sigma^c$ where $\Delta(X) = \sum_i X_i^{(1)} \otimes X_i^{(2)}$. Denote the antipode and the counit of $U_h\mathfrak{g}(\mathscr{E}, \Pi, p)^\sigma$ by S^σ and ε^σ respectively.

Let M_+ and M_- be subsets of $U_h\mathfrak{g}(\mathscr{E}, \Pi, p)^\sigma$ defined by $M_+ = \{E_{i_1}\cdots E_{i_x}H_{(1)}\cdots H_{(z)}\sigma^c \mid H_{(k)} \in \mathfrak{h}, x, z, c \in \boldsymbol{Z}_{\geq 0}\}$ and $M_- = \{H_{(1)}\cdots H_{(z)}\sigma^c F_{j_1}\cdots F_{j_y} \mid H_{(k)} \in \mathfrak{h}, y, z, c \in \boldsymbol{Z}_{\geq 0}\}$. Define $deg_+ : M_+ \to \boldsymbol{Z}$, $deg_- : M_- \to \boldsymbol{Z}$ by $deg_+(E_{i_1}\cdots E_{i_x}H_{(1)}\cdots H_{(z)}\sigma^c) = \chi + z$, $deg_-(H_{(1)}\cdots H_{(z)}\sigma^c F_{j_1}\cdots F_{j_y}) = z + y$. For $\varphi(h) \in \boldsymbol{C}[[h]] \setminus \{0\}$, put $\nu(\varphi(h)) = \lim_{h\to 0}(h(d\varphi(h)/dh)/\varphi(h))$. Let $U_h^{\sqrt{}}\mathfrak{b}_+^\sigma$ (resp. $U_h^{\sqrt{}}\mathfrak{b}_-^\sigma$) be a subset of $U_h\mathfrak{g}(\mathscr{E}, \Pi, p)^\sigma$ defined by $\{\sum_{i=1}^\infty \varphi_i(h)X_i$ $(X_i \in M_+$ (resp. $X_i \in M_-$), $\varphi_i(h) \in \boldsymbol{C}[[h]]) \mid \lim_{i\to\infty}(\nu(\varphi_i(h)) -$

$2^{-1}deg_+(X_i)) = +\infty$ (resp. $\lim_{i\to\infty}(\nu(\varphi_i(h)) - 2^{-1}deg_-(X_i)) = +\infty)\}$. Then $U_h^{\sqrt{}}\mathfrak{b}_+^\sigma$ and $U_h^{\sqrt{}}\mathfrak{b}_-^\sigma$ are Hopf subalgebras. Let $\boldsymbol{C}((h))$ be the quotient field of $\boldsymbol{C}[[h]]$.

Theorem C. *There is a non-degenerate $\boldsymbol{C}[[h]]$-bilinear form $\langle,\rangle : U_h^{\sqrt{}}\mathfrak{b}_+^\sigma \times U_h^{\sqrt{}}\mathfrak{b}_+^\sigma \to \boldsymbol{C}((h))$ determined by the conditions:*

(i) $\langle\sigma^c, H\rangle = 0$, $\langle\sigma^c, F_j\rangle = 0$, $\langle H, F_j\rangle = 0$, $\langle H, \sigma^d\rangle = 0$, $\langle E_j, \sigma^d\rangle = 0$, $\langle E_j, H\rangle = 0$, $\langle\sigma^c, \sigma^d\rangle = (-1)^{cd}$, $\langle H_\mu, H_\nu\rangle = -h^{-1}(\mu, \nu)$, $\langle E_i, F_j\rangle = (e^{-h} - e^h)^{-1}\delta_{ij}$.

(ii) $\langle X, xy\rangle = \langle\Delta^\sigma(X), x\otimes y\rangle$, $\langle XY, x\rangle = \langle Y\otimes X, \Delta^\sigma(x)\rangle$, $\langle S^\sigma(X), S^\sigma(x)\rangle = \langle X, x\rangle$, $\langle X, 1\rangle = \varepsilon^\sigma(X)$, $\langle 1, x\rangle = \varepsilon^\sigma(x)$.

Put $\bar{U}_h{}^\sigma = U_h\mathfrak{g}(\mathscr{E}, \Pi, p)^\sigma \otimes C((h))$. For $\gamma \in Q^+$, let U_γ^+ (resp. $U_{-\gamma}^-$) be the $\boldsymbol{C}[[h]]$-submodule of $U_h^{\sqrt{}}\mathfrak{b}_+^\sigma$ (resp. $U_h^{\sqrt{}}\mathfrak{b}_-^\sigma$) generated by the elements $E_{i_1}\cdots E_{i_r}$ (resp. $F_{i_1}\cdots F_{i_r}\sigma^{p(\gamma)}$) such that $\alpha_{i_1} + \cdots + \alpha_{i_r} = \gamma$. Let $\{e_{\gamma,i}^+ \in U_\gamma^+\}$ and $\{e_{\gamma,i}^- \in U_\gamma^-\}$ be bases such that $\langle e_{\gamma,i}^+, e_{\gamma,j}^-\rangle = 0$ if $i \neq j$. Let $C_\gamma = \sum_i \langle e_{\gamma,i}^+, e_{\gamma,i}^-\rangle^{-1} e_{\gamma,i}^+ \otimes e_{\gamma,i}^- \in \bar{U}_h{}^\sigma \hat{\otimes} \bar{U}_h{}^\sigma$. Put $t_0 = (\sum_{i=1}^N H_{\varepsilon_i}\otimes H_{\varepsilon_i}) + H_\delta \otimes H_{\Lambda_0} + H_{\Lambda_0}\otimes H_\delta$, $c_0 = 2^{-1}\sum_{a,b=0}^1 (-1)^{ab}\sigma^a\otimes\sigma^b$.

Let $\bar{U}_h^{z\sigma}$ denotes the z-adic completion of $\bar{U}_h{}^\sigma \otimes_C \boldsymbol{C}((z))$. For $\bar{\lambda} = (\lambda_0, \ldots, \lambda_n) \in (\boldsymbol{C}((z))^\times)^{n+1}$, define a Hopf algebra map $\rho_{\bar{\lambda}} : \bar{U}_h{}^\sigma \to \bar{U}_h^{z\sigma}$ by $\rho_{\bar{\lambda}}(\sigma) = \sigma$, $\rho_{\bar{\lambda}}(H) = H$, $\rho_{\bar{\lambda}}(E_i) = \lambda_i E_i$, $\rho_{\bar{\lambda}}(F_i) = \lambda_i^{-1}F_i$. We say $\lambda = (\lambda_0, \ldots, \lambda_n) > \bar{\mu} = (\mu_0, \ldots, \mu_n)$ if $\lambda_i/\mu_i \in z\boldsymbol{C}[[z]]$ for all i. For $\bar{\lambda} > \bar{\mu}$, we put $\tilde{R}(\bar{\lambda}, \bar{\mu}) = (\sum_{\gamma\in Q^+}\rho_{\bar{\lambda}}\otimes\rho_{\bar{\mu}}(C_\gamma))\cdot \mathrm{e}^{-ht_0}\cdot c_0 \in \bar{U}_h^{z\sigma}\hat{\otimes}\bar{U}_h^{z\sigma}$. Define $\tau : \bar{U}_h^{z\sigma}\hat{\otimes}\bar{U}_h^{z\sigma} \to \bar{U}_h^{z\sigma}\hat{\otimes}\bar{U}_h^{z\sigma}$ by $\tau(X\otimes Y) = Y\otimes Y$. Define $i_{12}, i_{13}, i_{23} : \bar{U}_h^{z\sigma}\hat{\otimes}\bar{U}_h^{z\sigma} \to \bar{U}_h^{z\sigma}\hat{\otimes}\bar{U}_h^{z\sigma}\hat{\otimes}\bar{U}_h^{z\sigma}$ by $i_{12}(X\otimes Y) = X\otimes Y\otimes 1$, $i_{13}(X\otimes Y) = X\otimes 1\otimes Y$, $i_{23}(X\otimes Y) = 1\otimes X\otimes Y$. Put $\tilde{R}(\bar{\lambda}, \bar{\mu})_{ab} = i_{ab}(\tilde{R}(\bar{\lambda}, \bar{\mu}))$. By using the quantum double construction, we have:

Proposition. *Let $\bar{\lambda}, \bar{\mu}, \bar{\nu} \in (\boldsymbol{C}((z))^\times)^{n+1}$ be such that $\bar{\lambda} > \bar{\mu} > \bar{\nu}$.* Then we have:

(i) *The inverse of $\tilde{R}(\bar{\lambda}, \bar{\mu})$ is given by $\tilde{R}(\bar{\lambda}, \bar{\mu})^{-1} = S^\sigma\otimes id(\tilde{R}(\bar{\lambda}, \bar{\mu}))$.*

(ii) $\tilde{R}(\bar{\lambda}, \bar{\mu})(\rho_{\bar{\lambda}}\otimes\rho_{\bar{\mu}}\Delta^\sigma(X))\tilde{R}(\bar{\lambda}, \bar{\mu})^{-1} = \rho_{\bar{\lambda}}\otimes\rho_{\bar{\mu}}(\tau(\Delta^\sigma(X)))\ (X \in \bar{U}_h^\sigma)$

(iii) $\sum_{\gamma\in Q^+}\rho_{\bar{\lambda}}\otimes\rho_{\bar{\mu}}\otimes\rho_{\bar{\nu}}(\Delta^\sigma\otimes id(C_\gamma \mathrm{e}^{-ht_0}\cdot c_0)) = \tilde{R}(\bar{\lambda}, \bar{\nu})_{13}\tilde{R}(\bar{\mu}, \bar{\nu})_{23}$,

$\sum_{\gamma\in Q^+}\rho_{\bar{\lambda}}\otimes\rho_{\bar{\mu}}\otimes\rho_{\bar{\nu}}(id\otimes\Delta^\sigma(C_\gamma \mathrm{e}^{-ht_0}\cdot c_0)) = \tilde{R}(\bar{\lambda}, \bar{\nu})_{13}\tilde{R}(\bar{\lambda}, \bar{\nu})_{12}$,

(iv) $\tilde{R}(\bar{\lambda}, \bar{\mu})$ *satisfies the Yang-Baxter equation*

$$\tilde{R}(\bar{\lambda}, \bar{\mu})_{12}\tilde{R}(\bar{\lambda}, \bar{\nu})_{13}\tilde{R}(\bar{\mu}, \bar{\nu})_{23} = \tilde{R}(\bar{\mu}, \bar{\nu})_{23}\tilde{R}(\bar{\lambda}, \bar{\nu})_{13}\tilde{R}(\bar{\lambda}, \bar{\mu})_{12}.$$

Remark 1. Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]/(\text{center})$. For $U_h\mathfrak{g}'^\sigma$, we can also obtain results similar to the ones of this note. As the image of $\tilde{R}(\bar{\lambda}, \bar{\mu})$ under the vector representation of $U_h\widehat{sl}(L\mid M)'^\sigma$, we can recover the Perk and Schultz $\boldsymbol{R}$-matrix [5], which is an extention (with $\boldsymbol{Z}/2\boldsymbol{Z}$-parameters) of Jimbo's $\boldsymbol{R}$-matrix obtained by using $U_h\widehat{sl}(L+M)'$.

Remark 2. Similarly to Tanisaki's argument [6], we can give analogues of the Casimir element and the Killing form for $U_h\mathfrak{g}(\mathscr{E}, \Pi, p)^\sigma$ by using $\langle,\rangle$.

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