71. *Convolution Semigroups of Stable Distributions over a Nilpotent Lie Group*

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We study stable properties of convolution semigroups of probability distributions over a Lie group. Stable distributions over a Heisenberg group or more generally on a homogeneous group were studied by Hulanicki [3], Glowacki [1] and others. Our stable distribution is motivated by these works. However, our definition is more general than their's, thereby including all strictly operator-stable distributions in case where the underlying group is a Euclidean space.

1. Convolution semigroup of probability distributions. Let G be a Lie group of dimension d. Elements of G are denoted by σ, τ etc. Let $\mathcal{G}$ be its left invariant Lie algebra, where an inner product $\langle\ ,\ \rangle$ and the associated norm $|\ |$ are defined, so that it can be identified with an Euclidean space $\boldsymbol{R}^d$. Elements of $\mathcal{G}$ are denoted by X, Y etc. We fix its basis $\{X_1, \dots, X_d\}$. Let C be the set of all continuous maps from the Lie group G into $\boldsymbol{R} = (-\infty, \infty)$ (such that $\lim_{\sigma\to\infty} f(\sigma)$ exists if G is non compact, where ∞ is the infinity). It is a Banach space by the supremum norm. We denote by C^2 the totality of $f \in C$ such that it is twice continuously differentiable and Xf, YZf belong to C for any X, Y, Z.

Let μ be a probability distribution over G. Let $\varphi : G \to G$ (or $G \to \mathcal{G}$ or $\mathcal{G} \to G$) be a continuous map. The transformation of μ by φ is defined by $\varphi\mu(A) = \mu(\varphi^{-1}(A))$. For two distributions μ and ν, their convolution is a distribution on G defined by $\mu * \nu(A) = \int_G \mu(d\sigma)\nu(\sigma^{-1}A)$. The n-ple convolution of the distribution μ is denoted by μ^{n*}.

A family of probability distributions $\{\mu_t\}_{t>0}$ over the Lie group G is called a *convolution semigroup* (*of probability distributions*), if it satisfies (i) $\mu_s * \mu_t = \mu_{s+t}$ for all s, $t > 0$, and (ii) μ_h converges weakly to δ_e as $h \to 0$, where δ_e is the unit measure at the unit element e of G.

Suppose that we are given a convolution semigroup of probability distributions $\{\mu_t\}_{t>0}$ over G. We set for $f \in C$, $T_t f(\tau) = \int_G f(\tau\sigma)\mu_t(d\sigma)$. Then $\{T_t\}_{t>0}$ defines a semigroup of strongly continuous linear operators on the Banach space C. The infinitesimal generator L of $\{T_t\}_{t>0}$ is often called the *infinitesimal generator of* $\{\mu_t\}_{t>0}$. Hunt [4] has shown that the domain of the infinitesimal generator L includes C^2 and represented Lf, $f \in C^2$ by making use of the basis of the Lie algebra $\mathcal{G}$ and a Lévy measure on the Lie group G. We shall obtain another representation of the infinitesimal generator in the case where the Lie group G is simply connected and nilpotent. An important fact on a simply connected nilpotent Lie group is that the exponential map $\exp : \mathscr{G} \to G$ is a diffeomorphism. See Hochschild [2].

Theorem 1.1. *Let L be the infinitesimal generator of a convolution semigroup of probability distributions $\{\mu_t\}_{t>0}$ over a Lie group G. If G is simply connected and nilpotent, there exists a symmetric nonnegative definite linear map $A = (a_{ij})$ on $\mathscr{G}$, a measure M on $\mathscr{G} - \{0\}$ with $\int \frac{|X|^2}{1+|X|^2} M(dX) < \infty$ and a vector $B = (b_i)$ on $\mathscr{G}$ such that Lf is represented by*

$$
(1.1) \quad Lf(\tau) = \frac{1}{2} \sum_{j,k} a_{jk} X_j X_k f(\tau) + \sum_j b_j X_j f(\tau) + \int_{\mathscr{G}-\{0\}} \left\{ f(\tau \exp X) - f(\tau) - \frac{1}{1+|X|^2} X f(\tau) \right\} M(dX)
$$

for any $f \in C^2$. Further, the triple (A, M, B) is uniquely determined by the convolution semigroup $\{\mu_t\}_{t>0}$.

Conversely suppose we are given a triple (A, M, B) over a Lie algebra $\mathscr{G}$ of a Lie group G, satisfying the above condition. Then there exists a unique convolution semigroup of probability distributions over G, whose infinitesimal generator is given by (1.1).

The triple (A, M, B) is called the *characteristics* of $\{\mu_t\}_{t>0}$.

Now let $\{\tilde{\mu}_t\}_{t>0}$ be a convolution semigroup of probability distributions over $\mathscr{G}$. Then its characteristic function $\phi_t(Z) = \int_{\mathscr{G}} \exp i\langle X, Z \rangle \tilde{\mu}_t(dX)$ is given by the Lévy-Khinchine formula.

$$
(1.2) \quad \phi_t(Z) = \exp \left[-\frac{1}{2} \langle Z, AZ \rangle + \int_{\mathscr{G}} \left(e^{i\langle Z, X \rangle} - 1 - \frac{i\langle Z, X \rangle}{1+|X|^2} \right) M(dX) + i\langle Z, B \rangle \right] t.
$$

The triple (A, M, B) is called the *characteristics* of $\{\tilde{\mu}_t\}_{t>0}$.

Theorem 1.2. *Let $\{\tilde{\mu}_t\}_{t>0}$ be a convolution semigroup of probability distributions over a Lie algebra $\mathscr{G}$ of a Lie group G with characteristics (A, M, B). Then*

$$
(1.3) \quad \mu_t = \lim_{n\to\infty} (\exp \tilde{\mu}_{t/n})^{n*}
$$

exists for all $t > 0$, where lim *is taken in the sense of the weak convergence. Further $\{\mu_t\}_{t>0}$ defines a convolution semigroup of probability distributions over the Lie group G with the characteristics (A, M, B).*

Conversely, let $\{\mu_t\}_{t>0}$ be a convolution semigroup of probability distributions over a Lie group G. If G is simply connected and nilpotent, there exists a unique convolution semigroup $\{\tilde{\mu}_t\}_{t>0}$ over the Lie algebra $\mathscr{G}$ satisfying (1.3) *for all $t > 0$. Its characteristics coinside with that of $\{\mu_t\}_{t>0}$.*

The convolution semigroup $\{\tilde{\mu}_t\}_{t>0}$ in Theorem 1.2 is called the *generating semigroup* of $\{\mu_t\}_{t>0}$.

Now we shall introduce a convolution semigroup of stable distributions. For this purpose we need some notations. Let $\{\gamma_r\}_{r>0}$ be a one parameter group of automorphisms of the Lie group G, i.e., (i) For each $r>0$, γ_r is a diffeomorphism G and satisfies $\gamma_r(\tau\sigma)=\gamma_r(\tau)\gamma_r(\sigma)$ for any $\tau, \sigma\in G$, (ii) $\gamma_r\gamma_s=\gamma_{rs}$ holds for any $r, s>0$, (iii) γ_r is continuous in $r\in(0,\infty)$. It is called a *dilation* if it satisfies (iv) $\gamma_r(\sigma)\to e$ uniformly on compact sets as $r\to 0$.

Let $d\gamma_r$ be the differential of the automorphism γ_r. Then $d\gamma_r$ defines an automorphism of $\mathcal{G}$ i.e., $d\gamma_r$ is a one to one linear map of $\mathcal{G}$ onto itself and satisfies $d\gamma_r[X, Y]=[d\gamma_r X, d\gamma_r Y]$ for any $X, Y\in\mathcal{G}$, where $[\ ,\ ]$ is the Lie bracket. Therefore $\{d\gamma_r\}_{r>0}$ is a one parameter group of automorphisms of $\mathcal{G}$. It satisfies $d\gamma_r X\to 0$ as $r\to 0$ for any $X\in\mathcal{G}$. The linear map $d\gamma_r$ is represented by $d\gamma_r=\exp(\log r)Q$, where Q is a linear map of $\mathcal{G}$ such that all of its eigen values have positive real parts. Further it satisfies $Q[X, Y]=[QX, Y]+[X, QY]$ for all $X, Y\in\mathcal{G}$. The map $d\gamma_r$ is often written as r^Q and the linear map Q is called the *exponent* of the dilation $\{\gamma_r\}_{r>0}$.

Remark. A dilation can not be defined on an arbitrary Lie group. Indeed if a dilation exists on the Lie group G, the Lie group is necessarily simply connected and nilpotent. See [7].

A convolution semigroup of probability distributions $\{\mu_t\}$ is called *stable with respect to a dilation* $\{\gamma_r\}_{r>0}$ if and only if $\gamma_r\mu_t=\mu_{rt}$ holds for any $r, t>0$.

In the case where G is a Euclidean space $\boldsymbol{R}^d$, a dilation $\{\gamma_r\}_{r>0}$ is nothing but a one parameter group of bijective linear transformations on $\boldsymbol{R}^d$ such that $\gamma_r x\to 0$ as $r\to 0$ for any $x\in\boldsymbol{R}^d$. If a convolution semigroup $\{\mu_t\}_{t>0}$ over $\boldsymbol{R}^d$ is stable with respect to a dilation $\{\gamma_r\}_{r>0}$, it is called *strictly operator-stable* (*with respect to the dilation* $\{\gamma_r\}_{r>0}$) according to Sharpe [9].

A convolution semigroup over a Lie algebra can be identified with a convolution semigroup over a Euclidean space. However, we emphasize that an arbitrary operator-stable convolution semigroup over a Euclidean space is not necessarily stable with respect to a certain dilation $\{\gamma'_r\}_{r>0}$ on the Lie algebra, because the dilation on the Lie algebra must satisfies the property $\gamma'_r[X, Y]=[\gamma'_r X, \gamma'_r Y]$ for all $X, Y\in\mathcal{G}$. For example, a convolution semigroup over $\boldsymbol{R}^d$ is always operator-stable if the Lévy measure M of the convolution semigroup is 0. However, regarding it as a convolution semigroup over a Lie algebra, it can be or can not be stable. It depends on the structure of the Lie algebra. Further discussions are given in [6].

Theorem 1.3. *Let $\{\mu_t\}_{t>0}$ be a convolution semigroup of probability distributions over a simply connected nilpotent Lie group G equipped with a dilation $\{\gamma_r\}_{r>0}$. Let $\{\tilde{\mu}_t\}_{t>0}$ be the associated generating convolution semigroup over the Lie algebra $\mathcal{G}$. Then $\{\mu_t\}_{t>0}$ is stable with respect to the dilation $\{\gamma_r\}_{r>0}$, if and only if $\{\tilde{\mu}_t\}_{t>0}$ is stable with respect to the dilation $\{d\gamma_r\}_{r>0}$.*

Proofs of Theorems 1.1, 1.2 and 1.3 are given in [6] in a different frame work, investigating Lévy processes on the Lie group G and the associated stochastic differential equations driven by Lévy processes with values in the Lie algebra $\mathcal{G}$.

2. Characterization of the infinitesimal generator of stable distributions. We shall characterize the stable property of the convolution semigroup by means of its infinitesimal generator. Somewhat different criteria for strictly operator stable semigroup over a Euclidean space are given in Sato [8] and Kunita [6].

Let G be a simply connected nilpotent Lie group equipped with a dilation $\{\gamma_r\}_{r>0}$. We need some facts on its exponent Q. Let g be the minimal polynomial of Q. It is factorized as $g = g_1^{n_1} \cdots g_p^{n_p}$, where $g_1, \ldots, g_p$ are distinct irreducible monic polynomials and n_j are positive integers. Set $W_j = \mathrm{Ker}(g_j(Q)^{n_j})$, $j = 1, \ldots, p$. These are Q-invariant subspaces of $\mathcal{G}$ and admits a direct sum decomposition $\mathcal{G} = \sum_j \oplus W_j$. Let $\kappa_j = \alpha_j \pm \sqrt{-1}\,\beta_j (\alpha_j, \beta_j$ are reals) be the roots of g_j (= eigen values of Q). We set

$$I = \{j \,;\, a_j = 1/2\},\ J = \{j \,;\, 1/2 < a_j < \infty\},\ I_1 = \{j \,;\, a_j = 1\},$$
$$J_1 = \{j \,;\, 1/2 < a_j < 1\}.$$

The subspaces of $\mathcal{G}$ are defined by $W_I = \oplus_{j \in I} W_j$ etc. and projectors to W_I, W_j etc. are denoted by T_{W_I}, T_{W_j} etc. We define $S = \{X \in \mathcal{G}\,;\, |X| = 1, |r^Q X| > 1$ for all $r > 1\}$. Then every $X \in \mathcal{G}(X \neq 0)$ is represented uniquely by $X = r^Q\theta$, where $r \in (0, \infty)$ and $\theta \in S$. We denote r and θ by $r(X)$ and $\theta(X)$.

In later discussions, the linear map $Q - I$ and its inverse plays an important role. If 1 is not an eigen value of Q, $Q - I$ is a bijection so that the inverse $(Q - I)^{-1}$ is well defined. Suppose that 1 is an eigen value of Q. We may assume that $\kappa_1 = 1$. Set $\tilde{W}_1 = \{(Q - I)X \,;\, X \in W_1\}$, $\hat{W}_1 = \{X \,;\, QX = X\}$ and $V = \oplus_{j \geq 2} W_j$, we choose a basis $\{Z_1, \ldots, Z_m, Y_1, \ldots, Y_n\}$ of W_1 such that $\hat{W}_1 = \{Z_1, \ldots, Z_m\}$ and $\tilde{W}_1 = \{(Q - I) Y_i \,;\, i = 1, \ldots, n\}$. Then we can define a linear map $(Q - I)^{-1} : \mathcal{G} \to V \oplus \{Y_1, \ldots, Y_m\}$ such that $(Q - I)^{-1}(Q - I) = T_{V \oplus \{Y_1, \ldots, Y_m\}}$. Indeed, since $(Q - I) : \{Y_1, \ldots, Y_m\} \to \tilde{W}_1$ and $(Q - I) : V \to V$ are bijections, the inverse $(Q - I)^{-1} : \tilde{W}_1 \oplus V \to \{Y_1, \ldots, Y_n\} \oplus V$ is well defined. For $X \in \hat{W}_1$ we set $(Q - I)^{-1}X = 0$.

Theorem 2.1. *Let $\{\mu_t\}_{t>0}$ be a convolution semigroup of probability distributions over a simply connected nilpotent Lie group G equipped with a dilation $\{\gamma_r\}_{r>0}$. It is stable with respect to the dilation if and only if its characteristics (A, M, B) admits the following properties.*

(i) *The linear map A satisfies $T_{W_I} A T'_{W_I} = A$ and $QA + AQ' = A$, where T'_{W_I}, Q' are the transposes of T_{W_I}, Q.*

(ii) *The measure M is supported by W_J. There exists a finite measure λ over S supported by $S_J \equiv S \cap W_J$ such that for any Borel subset E of W_J, M is represented by*

$$M(E) = \int_{S_J} \lambda(d\theta) \int_{(0,\infty)} \chi_E(r^Q\theta) r^{-2} dr. \tag{2.1}$$

(iii) (a) *If 1 is not an eigen value of Q, the vector B is determined by M and Q, and is given by the following B_1:*

$$B_1 = \int_{\mathcal{G}-\{0\}} \frac{2\langle QX, X\rangle}{(1 + |X|^2)^2} (Q - I)^{-1} X M(dX). \tag{2.2}$$

(b) *If 1 is an eigen value of Q, the measure M satisfies*

$$\int_{\mathcal{G}-\{0\}} \frac{2\langle QX, X\rangle}{(1 + |X|^2)^2} T_{W_1} X M(dX) \in \tilde{W}_1. \tag{2.3}$$

Further the vector B is given by $B_1 + B_0$, where B_0 is an element of $\hat{W}_1$.

Proof. Suppose first that the convolution semigroup is stable with respect to the dilation $\{\gamma_r\}_{r>0}$. Then its generating convolution semigroup $\{\tilde{\mu}_t\}_{t>0}$ is strictly operator stable with respect to the dilation $\{r^Q\}_{r>0}$. Hence $\tilde{\mu}_r = r^Q \tilde{\mu}_1$ holds for all $r > 0$. Then the characteristic function $\psi_r(Z)$ of $\tilde{\mu}_r$ is equal to $\psi_1(r^{Q'}Z)$ and is represented by

$$(2.4)\quad \exp\Big[-\frac{1}{2}\langle Z, r^Q A r^{Q'} Z\rangle + \int\Big(e^{i\langle Z,X\rangle} - 1 - \frac{i\langle Z, X\rangle}{1+|r^{-Q}X|^2}\Big) r^Q M(dX) + i\langle Z, r^Q B\rangle\Big],$$

where $r^Q M$ is the measure defined by $r^Q M(E) = M(r^{-Q}E)$ for all Borel sets E. Compare this with the characteristic function (1.2). Then we have $rA = r^Q A r^{Q'}$, $rM = r^Q M$ and

$$(2.5)\qquad (r^Q - r)B = r\int\Big(\frac{X}{1+|r^{-Q}X|^2} - \frac{X}{1+|X|^2}\Big) M(dX).$$

The first two equalities imply the assertions (i) and (ii) by Proposition 4.3.3 in [5] and Theorem 1.3 in [6]. We shall prove (iii). Divide both sides of the above by r and then differentiate them with respect to r. Then we obtain

$$(2.6)\qquad (Q - I)r^{Q-2}B = \int \frac{2\langle Qr^{-Q-1}X, r^{-Q}X\rangle X}{(1+|r^{-Q}X|^2)^2} M(dX).$$

Setting $r = 1$, we obtain

$$(2.7)\qquad (Q - I)B = \int \frac{2\langle QX, X\rangle X}{(1+|X|^2)^2} M(dX).$$

This implies (iii) immediately.

Conversely suppose that we are given an arbitrary triple (A, M, B) satisfying (i)-(iii). Then there exists a convolution semigroup $\{\tilde{\mu}_t\}_{t>0}$ of probability distributions over $\mathcal{G}$ with characteristics (A, M, B). We will show that it is strictly operator stable with respect to the dilation $\{r^Q\}_{r>0}$. The linear map A satisfies $rA = r^Q A r^{Q'}$ for all $r > 0$ in view of (i) and the measure M defined by (2.1) satisfies $rM = r^Q M$ for all $r > 0$. See eg. [5]. Further, the vector B satisfies (2.7) in both cases (a), (b). We shall prove that (2.7) imples (2.5). Note the relation $r^{-1}M = r^{-Q}M$. Then (2.7) implies

$$(Q - I)B = r\int \frac{2\langle QX, X\rangle X}{(1+|X|^2)^2} r^{-Q}M(dX) = r\int \frac{2\langle Qr^{-Q}X, r^{-Q}X\rangle r^{-Q}X}{(1+|r^{-Q}X|^2)^2} M(dX),$$

which is equivalent to (2.6). Integrating both sides of (2.6) with respect to r, and multiplying both sides by $r > 0$, we obtain (2.5). Now these three properties of (A, M, B) implies that the characteristic function $\psi_t(Z)$ of $\tilde{\mu}_t$ satisfies $\psi_r(Z) = \psi_1(r^{Q'}Z)$ for all $Z \in \mathcal{G}$ and $r > 0$. Therefore we have $\tilde{\mu}_r = r^Q \tilde{\mu}_1$ for all $r > 0$, proving that $\{\tilde{\mu}_t\}_{t>0}$ is strictly operator stable with repect to the dilation $\{r^Q\}_{r>0}$. Let $\{\mu_t\}_{t>0}$ be the convolution semigroup generated by $\{\tilde{\mu}_t\}_{t>0}$. It is stable with respect to the dilation $\{\gamma_r\}_{r>0}$ with characteristics (A, M, B) by Theorem 1.3. The proof is complete.

Corollary 2.2 (*cf. Kunita* [6]). *Let L be the infinitesimal generator of a convolution semigroup $\{\mu_t\}_{t>0}$ of probability distributions over a simply connected nilpotent Lie group G equipped with a dilation $\{\gamma_r\}_{r>0}$ with the exponent Q.*

(a) *Suppose that* 1 *is not an eigen value of the exponent* Q. *Then* $\{\mu_t\}_{t>0}$ *is stable with respect to the dilation if and only if* Lf, $f \in C^2$ *is represented by*

$$(2.8)\quad Lf(\tau) = \frac{1}{2}\sum_{j,k} a_{jk}X_jX_kf(\tau) + \int_S (Q - I)^{-1}T_{W_{I_1}}\theta f(\tau)\lambda(d\theta)$$
$$+ \int_{\mathscr{G}-\{0\}} (f(\tau \exp X) - f(\tau) - T_{W_{J_1}}Xf(\tau) - \chi(r(X) < 1)T_{W_{I_1}}Xf(\tau))M(dX),$$

where $A = (a_{jk})$ *and* M *satisfy* (i), (ii) *of Theorem* 2.1. *In particular,* $(a_{jk}) = 0$ *holds if* $I = \emptyset$, *and* $M = 0$ *holds if* $J = \emptyset$ *in* (2.8). *Further* $T_{W_{I_1}} = 0$ *holds if* $I_1 = \emptyset$, *and* $T_{W_{J_1}} = 0$ *holds if* $J_1 = \emptyset$ *in* (2.8).

(b) *Suppose that* 1 *is an eigen value of the exponent* Q. *Then* Lf, $f \in C^2$ *has an additional drift term* $B_0 f$ *in* (2.8), *where* $B_0 \in \hat{W}_1$. *Further the measure* λ *satisfies*:

$$(2.9)\qquad \int_S T_{W_1}\theta\lambda(d\theta) \in \tilde{W}_1.$$

In particular $\int_S T_{W_1}\theta\lambda(d\theta) = 0$ *holds if* $W_1 = \hat{W}_1$.

Proof. The representation (2.8) of the infinitesimal generator is immediate from Theorems 1.1 and 2.1, since the following (2.10)-(2.12) are satisfied.

$$(2.10)\quad T_{W_j}B_1 = \int_{\mathscr{G}-\{0\}} \frac{1}{1+|X|^2} T_{W_j}XM(dX) \text{ if } \alpha_j > 1,$$

$$(2.11)\quad = \int_{\mathscr{G}-\{0\}} \frac{|X|^2}{1+|X|^2} T_{W_j}XM(dX) \text{ if } 1/2 < \alpha_j < 1,$$

$$(2.12)\quad = \int_{\{r(X)\geq 1\}} \frac{1}{1+|X|^2} T_{W_j}XM(dX) - \int_{\{0<r(X)<1\}} \frac{|X|^2}{1+|X|^2} T_{W_j}XM(dX)$$
$$+ \int_S (Q-1)^{-1}T_{W_j}\theta\lambda(d\theta) \text{ if } \alpha_j = 1.$$

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