2. Galois Subfields of Abelian Function Field of Two Variables

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Let L be an abelian function field of two variables over C, and K be a Galois subfield of L, i.e., L is a finite algebraic Galois extension of K. We classify such K by a suitable complex representation of the Galois group $G = Gal(L/K)$.

Let A be the abelian surface with the function field L. Since $g \in G$ induces an automorphism of A, we have a complex representation $gz = M(g)z + t(g)$, where $M(g) \in GL_2(C)$, $z \in C^2$, and $t(g) \in C^2$. Fixing the representation, we put $G_0 = \{g \in G \mid M(g) = 1_2\}$, $H = \{M(g) \mid g \in G\}$ and $H_1 = \{M(g) \in H \mid det\, M(g) = 1\}$. Then we have the following exact sequences of groups:

$$1 \to G_0 \to G \to H \to 1,$$
$$1 \to H_1 \to H \xrightarrow{d} C_n \to 1,$$

where $d(M(g)) = detM(g)$, and H/H_1 is a cyclic group C_n of order $n \leq 12$. The quotient surface A/G_0 is also an abelian surface. Note that the function field of the surface A/G is isomorphic to K.

Definition. We call H a holonomy part of the complex representation of G.

The holonomy part is completely determined by Fujiki [1], in which he studies automorphisms fixing the origin. By a slightly different method from his, i.e., by considering Sylow groups of H, we can show the following readily.

Proposition 1. *The order of H is 5, 10 or $2^a \cdot 3^b$, where $a \leq 5$ and $b \leq 2$.*

Since the commutative group G_0 is a normal subgroup of G, we have

Corollary 2. *The Galois group G is solvable.*

The main purpose of this note is to classify K by using the holonomy part. But we have no suitable language in the category of fields, so we do the classification in the equivalent category, i.e., using the language of the birational classification of algebraic surfaces. Note that in the case of elliptic curve E the similar classification is simple, i.e., E/G is rational if and only if H is not trivial.

Let $[x, y]$ denote the diagonal matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ and e_n denote $exp(2\pi\sqrt{-1}/n)$. Then we have

Lemma 3. *If H contains $[e_n, e_n]$, where $n = 3, 4, 6$, then A is isomorphic to $E \times E$, where $E = C/(1, e_n)$.*

Since each $M \in H$ defines also an automorphism of A, the quotient spaces $X = A/G$ and $X' = A/H$ are normal algebraic surfaces. Now, let $\tilde{Y}$ be a nonsingular model of an algebraic surface Y. Then, let $q(Y)$ and $P_m(Y)$ denote the irregularity and the m-genus of $\tilde{Y}$ respectively.

Lemma 4. *We have that* $q(X) = q(X')$, $P_1(X) = P_1(X')$ *and* $P_m(X) \geq P_m(X')$.

Let S be a (relatively) minimal model of X and $F(G)$ denote the set of fixed points of G. Then the result is stated as follows:

Theorem 5. *We have the following classification table.*

H	structure of S		
$= \{1_2\}$	abelian surface		
$\neq \{1_2\} = H_1$	$K3$ surface		
$\neq H_1$	$H = \langle [1, e_n] \rangle$	$F(G) = \emptyset$	hyperelliptic surface
		$F(G) \neq \emptyset$	ruled surface with $q = 1$
	$H = \langle [1, -1], [-1, 1] \rangle$	$F(G) =$ finite	Enriques surface
		$F(G) \supset$ *curve*	rational surface
	except the above	rational surface	

In this table $n = 2,3,4,$ *or* 6.

Corollary 6. *If the order of* $H > 24$ *or the degree of the eigenvalue(s) of* M *is* 4, *then* S *is rational.*

When the degree of the eigenvalue of M is 4, A is isogenous to $A(n)$, which is defined as follows (cf. [5]): let $\zeta = e_n$, $n = 5,8,10$ *or* 12. Put

$$\Omega_n = \begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^k & \zeta^{2k} & \zeta^{3k} \end{pmatrix},$$

where $k = 2,3,3,5$ corresponding to value n respectively. Then $A(n) = \boldsymbol{C}^2/\Omega_n$ is an abelian surface and $M = [\zeta, \zeta^k]$ defines an automorphism of $A(n)$. Furthermore in the case when $n = 5$, A is isomorphic to $A(5)$, which is a simple abelian surface and is the Jacobian variety of the curve $y^2 = x^5 + 1$. Looking at the tables in Fujiki [1], we notice that S is rational in many cases. But if A is simple (and S is rational), then there exists only one abelian surface $A \cong A(5)$.

If K is rational, then the order of $G \geq 3$. We know that there is only one abelian surface if K is rational and the order of G is 3. In this case A must be $E \times E$, where $E = \boldsymbol{C}/(1, e_3)$ and $H = \langle [e_3, e_3] \rangle$ (cf. [4]). Hence we have that $d_r(E \times E) = 3$, where d_r is the degree of irrationality (see, [6]). Except this case it seems difficult to find the value $d_r(A)$ even in the case when K is rational. So we ask the following

Question 7. Find the value $d_r(A)$.

Example 8. Note that the holonomy part which defines an Enriques surface is unique. An example of G is as follows: let $A = E_1 \times E_2$ and $E_i = \boldsymbol{C}/(1, \tau_i)$, $Im\tau_i > 0$ and $G = \langle g_1, g_2 \rangle$, where

$$\begin{cases} g_1 z = [-1, -1]z, \text{ and} \\ g_2 z = [1, -1]z + {}^t(1/2, 0). \end{cases}$$

Note 9. In general $P_m(A/G)$ and $P_m(A/H)$ are distinct from each other. In fact, if $H = \langle [1, e_n] \rangle$ and $F(G) = \emptyset$, then A/G and A/H are hyperelliptic and ruled surfaces respectively. All the G which define hyperelliptic surfaces are given in Suwa [3].

The proof of our results depends in many parts on the work of Katsura [2]. Details will appear elsewhere.

References

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