70. *Note on Global Existence for Axially Symmetric Solutions of the Euler System*

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1. Consider axially symmetric solutions of the 3-D Euler system

(1.1) $\partial_t u + (u\cdot\nabla)u + \nabla p = 0, \quad \nabla\cdot u = 0, \quad \text{in } [0,\infty)\times\Omega,$

(1.2) $u\cdot n = 0, \qquad \text{on } [0,\infty)\times\partial\Omega,$

(1.3) $u(0,x) = u_0(x), \qquad \text{for } x\in\Omega.$

Here Ω is assumed to be $\boldsymbol{R}^3$, or a bounded domain with sufficiently smooth boundary which is obtained by rotation about the x_3-axis of a simply connected planar domain lying in the half plane $\boldsymbol{R}^2_+ = \{x = (x_1, x_2, x_3) \mid x_2 = 0, x_1 \geq 0\}$; $u = u(t,x) = (u_1, u_2, u_3)$ is the velocity, $p = p(t,x)$ the pressure; $u_0(x)$ is the axially symmetric initial velocity.

In cylindrical coordinates r, θ, z, the velocity u is written as $u = \alpha e_r + \beta e_\theta + \gamma e_z$ where $e_r = (\cos\theta, \sin\theta, 0)$, $e_\theta = (-\sin\theta, \cos\theta, 0)$, and $e_z = (0,0,1)$. Axial symmetry means that $\beta = 0$ and α, γ are independent of θ.

The global in time existence of axially symmetric solutions to (1.1)-(1,3) with $\Omega = \boldsymbol{R}^3$ was proven by authors Ukhovski and Iudovich [6], Maida [3], Raymond [4]. In these papers, they have used the assumption, among others, that $(\mathrm{rot}\, u_0)r^{-1} \in L^\infty(\boldsymbol{R}^3)$, which is automatically satisfied when $u_0 \in H^s(\boldsymbol{R}^3)$ $(s > 7/2)$.

In the present paper, we will develop our approach for axially symmetric solutions, by proving the following theorem, where the above assumption is superfluous even if $u_0 \in H^s(\boldsymbol{R}^3)$ $(5/2 < s \leq 7/2)$. Here we denote by $H^s(\Omega)$ the Sobolev space of order s on Ω, and also denote by $C^j(I;X)$ the set of functions $u = u(t,x)$ such that $\partial_t^i u$, $0 \leq i \leq j$, are X-valued continuous on the interval I.

Theorem. *Assume that Ω is $\boldsymbol{R}^3$ or a bounded domain with the property mentioned above. Suppose that $u_0 \in H^s(\Omega)$ is axially symmetric and satisfy that $u_0\cdot n = 0$ on $\partial\Omega$ and $\nabla\cdot u_0 = 0$ in Ω. Then there exists a unique axially symmetric solution u of* (1.1)-(1.3) *such that $u \in C^0([0,\infty); H^s(\Omega)) \cap C^1([0,\infty); H^{s-1}(\Omega))$. Here we assume that $s > 5/2$ for the former case or $s \geq 3$ (integer) for the latter case respectively.*

Our devices such as (3.1) and (3.2) below are also useful to show the corresponding results to the same system with the wider classes of initial data.

2. In cylindrical coordinates, the vorticity ξ has the form

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(2.1) $$\xi = \nabla \times u = \omega e_\theta, \quad \text{where } \omega = \partial_z \alpha - \partial_r \gamma,$$
and from (1.1) we see that ω is governed formally by the equation
(2.2) $$\partial_t \omega + \alpha \partial_r \omega + \gamma \partial_z \omega - r^{-1} \alpha \omega = 0.$$
We remark that (2.2) become $(D/Dt)(\omega/r) = 0$, where $D/Dt = \partial_t + \alpha \partial_r + \gamma \partial_z$.

Now we assume that there exists an axially symmetric solution u of (1.1)-(1.3) such that $u \in C^0([0, T); H^s(\Omega)) \cap C^1([0, T); H^{s-1}(\Omega))$ for a given constant $T > 0$. Then, by virtue of the continuation principle given in [1], [2] and [5] to prove the Theorem stated above, we have only to show that
(2.3) $$\int_0^T \| \omega(\tau) \|_{L^\infty(\Omega)} d\tau < \infty.$$

Hereafter in this section we shall show certain properties of u in the cylindrical coordinates. Fix $t \in [0, T)$. Then, since $u(t) = (\alpha(t) \cos\theta, a(t) \sin\theta, \gamma(t)) \in B^1(\bar{\Omega})$, we see that
(2.4) $$\alpha(t), \gamma(t) \in B^1(\bar{D}), \quad \alpha(t) r^{-1} \in B^0(\bar{D}).$$
Here D is a planar domain such that $D = \Omega \cap \boldsymbol{R}^2_+$ and $B^i(\bar{D})$ is the space of functions whose j-th derivatives, $j \leq i$, are continuous and bounded on $\bar{D}$. We remark that $\partial_{x_1} u_1 + \partial_{x_2} u_2 = \partial_r \alpha + \alpha r^{-1}$, $\partial_{x_2} u_1 = (\partial_r \alpha - \alpha r^{-1}) \sin\theta \cdot \cos\theta$. Furthermore by (1.2) and the axial symmetry of the solution u, it holds that
(2.5) $$\alpha(t) n_r + \gamma(t) n_z = 0 \text{ on } \partial D,$$
where (n_r, n_z) is the unit outer normal on ∂D if ∂D has no corners. If ∂D has corners, we see that $\alpha = \gamma = 0$ there.

Now we use the particle trajectory transformation $(\tilde{r}(t; r, z), \tilde{z}(t; r, z))$ defined as the solution of
$$(d/dt)\tilde{r}(t; r, z) = \alpha(t, \tilde{r}(t; r, z), \tilde{z}(t; r, z)),$$
(2.6) $$(d/dt)\tilde{z}(t; r, z) = \gamma(t, \tilde{r}(t; r, z), \tilde{z}(t; r, z))$$
$$\text{with } (\tilde{r}(0; r, z), \tilde{z}(0; r, z)) = (r, z) \text{ for } (r, z) \in \bar{D}.$$
Here and hereafter, we use the notation such that $\alpha(t, X) = \alpha(t, r, z)$ for any $X = (r\cos\theta, r\sin\theta, z)$ because of the axis symmetry of $u(t, x)$.

Then from (2.4), (2.5), and the incompressibility of u $(r^{-1}\partial_r(r\alpha) + \partial_z \gamma = 0)$, we see that for a fixed $t \in [0, T)$
(2.7) $(\tilde{r}(t; r, z), \tilde{z}(t; r, z))$ is a diffeomorphism of $\bar{D}$ and
(2.8) it preserves the measure $rdrdz$ on D.

3. To obtain (2.3), we need the following lemmas which are proved in sections 4 and 5.

Lemma 1. *For any $\delta > 0$, there exists a positive constant k depending only on Ω such that, for $X = (r, 0, z) \in D$ and $t \in [0, T)$,*
(3.1) $$|\alpha(t, X)| \leq k\Big(\int_{\substack{|X-Y| \leq \delta \\ Y \in \Omega}} |\omega(t, Y)| \, |X - Y|^{-2} dY + r \int_{\substack{|X-Y| > \delta \\ Y \in \Omega}} |\omega(t, Y)| \, |X - Y|^{-3} dY\Big).$$

From Lemma 1 and (2.2), (2.8) we have

Lemma 2. *Let s', s'' be constants such that $2 < s'' < 3 < s' < 5$ and*

$\omega_0/r \in L^{s'}(D) \cap L^{s''}(D)$. *Then there exists a constant* k *depending only on* $\|\omega_0/r\|_{L^{s'}(D)}$, $\|\omega_0/r\|_{L^{s''}(D)}$ *such that the following estimate holds*:

(3.2) $\quad |\alpha(t, r, z)| \leq kr$ *for any* $(r, z) \in D$, $r > 0$ *and* $t \in [0, T)$,

where $\omega_0 = rot\, u_0$.

(Hereafter we use k to denote constants having the same dependence cited above.)

Not, since $\xi_0 = \omega_0 e_\theta \in H^{s-1}(\Omega)$ and since for $\xi = \omega e_\theta$, $\partial_{x_1}\xi_2 - \partial_{x_2}\xi_1 = \partial_r\omega + \omega r^{-1}$, $\partial_{x_1}\xi_1 = -(\partial_r\omega - \omega r^{-1})\sin\theta\cos\theta$, the imbedding theorem for the Sobolev spaces $(H^{s-2}(\Omega) \subset L^q$ if $q \geq 2$ and $s-2 > 3/2 - 3/q)$ implies that the assumption of Lemma 2 is satisfied, for $s > 3/2 + 1$. Therefore from Lemma 2 and (2.6), we see that

$$(\tilde{r}(t\,; r, z) \leq re^{kt} \quad \text{for } (r,z) \in D,\ r>0 \text{ and } t \in [0, T).$$

Thus by the remark cited under (2.2) and (2.7) we have

$$\|\omega(t)\|_{L^\infty(\Omega)} \leq e^{kt}\|\omega(0)\|_{L^\infty(\Omega)} \text{ for } t \in [0, T)$$

so that the desired boundedness (2.3) of ω is shown.

4. Proof of Lemma 1. we first consider the case where $\Omega = \boldsymbol{R}^3$. Since $u(t) \in L^2(\Omega)$ and $\xi(t) \in B^0(\bar{D}) \cap L^2(\Omega)$, by the Biot-Savart law, we have that for $0 \leq t < T$

$$(4.1) \quad \alpha(t, X) = (4\pi)^{-1}\int_{\boldsymbol{R}^3}(z-z')\,|X-Y|^{-3}\cos(\theta-\theta')\omega(t, Y)dY,$$

where $X = (r\cos\theta, r\sin\theta, Z)$ and $Y = (r'\cos\theta', r'\sin\theta', Z')$.

Let X be $(r, 0, z)$. Then by the change of independent variables such as $\theta' \to \theta' + \pi$ for $\theta' \in (\pi/2, 3\pi/2)$ we have

$$\begin{aligned}\alpha(\cdot, r, z) = (4\pi)^{-1}\int_0^\infty\int_{-\infty}^\infty\int_{-\pi/2}^{\pi/2} &\{(r^2 + r'^2 - 2rr'\cos\theta' + |z-z'|^2)^{-3/2} \\ &- (r^2 + r'^2 + 2rr'\cos\theta' + |z-z'|^2)^{-3/2}\} \\ &\times (z-z')\cos\theta'\omega(\cdot, r', z')r'dr'dz'd\theta'.\end{aligned} \tag{4.2}$$

Let $\delta > 0$ be a constant. Fix the point $X = (r, 0, z)$. We denote by C the circle $\{(r'\cos\theta', r'\sin\theta', z') \mid 0 \leq \theta' < 2\pi)$ for each (r', z'). Then we divide the configurations of the point X and the circles C into the following three cases:

α) The case where there exist two angles θ^+, θ^- such that $0 < \theta^+ < \pi$, $\pi < \theta^- < 2\pi$ and $r^2 + r'^2 - \cos 2\,\theta^\pm rr' + |z-z'|^2 = \delta^2$. Then we define $C^+ = \{Y \in C \mid |X-Y| > \delta\}$ and $C^- = \{Y \in C \mid |X-Y| < \delta\}$. That is, $C^+ = \{r'\cos\theta, r'\sin\theta', z') \mid \theta^+ < \theta' < \theta^-\}$ and $C^- = \{r'\cos\theta, r'\sin\theta', z') \mid 0 \leq \theta' < \theta^+, \theta^- < \theta' < 2\pi\}$.

β) The case where $|X-Y| \geq \delta$ for any $Y \in C$.

γ) The case where $|X-Y| \leq \delta$ for any $Y \in C$.

Setting $\hat{Y} = (r'\cos(\pi+\theta'), r'\sin(\pi+\theta'), z')$ for $-\pi/2 < \theta' < \pi/2$, $\hat{X} = (-r, 0, z)$, we observe that $|X-\hat{Y}| = |\hat{X}-Y|$ and $\||X-Y|^{-3} - |\hat{X}-Y|^{-3}| \leq 3|X-\hat{X}|\,|X-Y|^{-4}$. Hence, in the case β), it follows from (4.2) that

$$I(C) \equiv \left|\int_C (r^2 + r'^2 - 2rr'\cos\theta' + |z-z'|^2)^{-3/2}(z-z')\cos\theta' d\theta'\right|$$

$$\leq 6r\int_{-\pi/2}^{\pi/2}|X-Y|^{-3}d\theta'.$$

In the case γ), we can easily see that $I(C)$ above is estimated by $\int_C |X-Y|^{-2}d\theta'$. In the case α), since it is readily shown that $I(C^-) \leq \int_{C-}|X-Y|^{-2}d\theta'$, it remains to estimate $I(C^+)$. The $I(C^+)$ when $\theta^+ \geq \pi/2$ and the $I(C')$ when $\theta^+ < \pi/2$ are estimated by $\int_{C-}|X-Y|^{-2}d\theta'$, because $|X-\hat{Y}| \geq |X-Y|$ for $-\pi/2 < \theta' < \pi/2$. Here $C' = \{\pi - \theta' \mid \theta' \in C^-\}$. On the other hand, $I(C^+ \setminus C')$ is estimated by the same way as in the case β). Now it is not difficult to verify the estimate (3.1) in which k depends only on π.

Next we give the proof of (3.1) when Ω is the bounded domain.

Incompressibility under axial symmetry (i.e., $r^{-1}\partial_r(r\alpha) + \partial_r\gamma = 0$) ensures to exist a stream function ψ such for a fixed t

(4.3) $$\alpha = -\partial_z\psi,\ r\gamma = \partial_r(r\psi) \text{ in } D$$

since D is assumed to be simply connected, and from (2.1), (2.4) it follows that

(4.4) $$L\psi = -\omega \text{ in } D, \text{ where } L\psi = \partial_r(r^{-1}\partial_r(r\psi)) + \partial_z^2\psi.$$

Here by using the curilinear integral along the path appropriately and from (2.4), (2.5), (4.3) we see that

$$\psi(t, r, z) = 0 \text{ on } \partial D,\ \psi,\ \partial_r\psi,\ \psi/r,\ \partial_z\psi \in B^0(\bar{D}).$$

Therefore we obtain that

$$\psi e_\theta \in H_0^1(\Omega),\ \partial_r\psi/r,\ \psi/r^2 \in L^1(\Omega)$$

which, together with (2.4), (4.3) (4.4), imply that

$$(\omega e_\theta, v)_{L^2(\Omega)} = (\nabla(\psi e_\theta), \nabla v)_{L^2(\Omega)}$$
$$= (\partial_r(\psi e_\theta), \partial_r v)_{L^2(\Omega)} + (\partial_z(\psi e_\theta), \partial_z v)_{L^2(\Omega)} + (r^{-1}\partial_\theta(\psi e_\theta), r^{-1}\partial_\theta v)_{L^2(\Omega)}$$

for any vector $v \in \mathcal{D}(\Omega)$, hence for any vector $v \in H_0^1(\Omega)$. Thus, noting the fact that $\omega e_\theta \in H^{s-1}(\Omega) \subset C^r(\bar{\Omega})$ with $0 < \gamma < 1$, we have the following representation of α in Ω corresponding to (4.1):

$$\alpha(t, X) = -\partial_z\psi(t, r, z) = \int_\Omega \partial_z K(X, Y)\cos(\theta - \theta')\omega(t, Y)dY,$$

for $X = (r\cos\theta, r\sin\theta, z)$ with an arbitrary θ.

Here $K(X, Y)$ is the (Dirichlet) Green function of Laplacian on Ω satisfying that $|D_X^\beta K(X, Y)| \leq c_\beta |X - y|^{-1-|\beta|}$ for any multi-index β and $X, Y \in \bar{\Omega}(X \neq Y)$, where c_β is a constant depending only on β and Ω. Furthermore, introducing the rotation R_φ defined by $(R_\varphi f)(X) = f(r\cos(\theta+\varphi), r\sin(\theta+\varphi), z)$ for $X = (r\sin\theta, r\cos\theta, z) \in \Omega$, we see that $(R_\varphi \Delta u)(X) = \Delta(R_\varphi u)(X)$ for any φ. Hence, we obtain that $K(R_\varphi X, Y) = K(X, R_{-\varphi}Y)$ for φ and $X, Y \in \bar{\Omega}(X \neq Y)$. Here, of course, $R_\varphi X$ means $(R_\varphi id)(X)$.

Therefore, corresponding to the derivation of (4.2) from (4.1), we have

$$\int_0^{2\pi} \partial_z K((r, 0, z), Y)\cos\theta' d\theta'$$

$$= \int_{-\pi/2}^{\pi/2} \{\partial_z K((r,\, 0,\, z),\, Y)$$

$$- \partial_z K((r,\, 0,\, z),\, (r' \cos(\pi + \theta'),\, r' \sin(\pi + \theta'),\, z'))\} \cos \theta' \, d\theta'$$

$$= \int_{-\pi/2}^{\pi/2} \{\partial_z K((r,\, 0,\, z),\, Y) - \partial_z K((-\, r,\, 0,\, z),\, Y)\} \cos \theta' \, d\theta'$$

$$= \int_{0}^{\pi/2} \left\{\int_0^1 (d/dt) \partial_z K((r \cos \theta^+(t),\, r \sin \theta^+(t),\, z),\, Y) dt\right\} \cos \theta' \, d\theta'$$

$$+ \int_{-\pi/2}^{0} \left\{\int_0^1 (d/dt) \partial_z K((r \cos \theta^-(t),\, r \sin \theta^-(t),\, z),\, Y) dt\right\} \cos \theta' \, d\theta',$$

where $\theta^{\pm}(t) = \pm\, \pi(t-1)$ for $0 \leq t \leq 1$. Then we can get the estimate (3.1) for the bounded domain Ω, by applying the same procedure as in the previous case to the above integrals, and by using the estimates of $|\, D_X^{\beta} K(X, Y)\, |$ for $|\, \beta\, | = 1{,}2$, cited above. Therefore in this case the constant k in (3.1) depends only on c_{β} for $|\, \beta\, | = 1{,}2$.

5. Proof of Lemma 2. From (2.2), (2,4), (2.8) (i.e. $\nabla \cdot u = 0$) and by a certain limit process we have that for $q = s'$ or s'' in Lemma 2, $t \in [0,\, T)$,

$$\| (\omega / r)(t,\, r,\, z) \|_{L^q(D)} \leq \| (\omega_0 / r)(r,\, z) \|_{L^q(D)}. \tag{5.1}$$

Now let us denote the right-hand side of (3.1) by $k \times (I_1 + I_2)$. Then, by taking $\delta = \min(r,\, 1)$ for the term I_1, we see that $r' \leq 2r$. So, by (5.1) we have

$$I_1 \leq \text{const.}\ r\left(\int_0^{\delta} R^{-2q+2} dR\right)^{1/q} \| (\omega_0 / r)(r,\, z) \|_{L^{q'}(D)} \leq kr. \tag{5.2}$$

Here we have taken q, q' such that $1/q + 1/q' = 1$, and $5/4 < q < 3/2$, $q' = s' > 3$. Next we turn to the estimation of I_2. By using (5.1) again, we have

$$I_2 = r \int_{\substack{|X-Y| \geq \delta \\ Y \in \Omega}} (|\, \omega(t,\, Y)\, | / r') r' \,|\, X - Y\, |^{-3} dY$$

$$\leq \text{const.}\ r \| (\omega_0 / r) \|_{L^{p'}(D)} \cdot \left(\int_{|Y| \geq \delta} (r + r')^p \,|\, Y\, |^{-3p} dY\right)^{1/p} \tag{5.3}$$

$$\leq kr\left\{\left(\int_{|Y| \geq \delta} r^p \,|\, Y\, |^{-3p} dY\right)^{1/p} + \left(\int_{|Y| \geq \delta} |\, Y\, |^{-2p} dY\right)^{1/p}\right\}$$

$$\leq kr(r + \delta)\delta^{-3(p-1)/p}.$$

Here we have chosen p, p' such that $1/p + 1/p' = 1$ and $3/2 < p < 2$, $p' = s'' < 3$. Note that $1 - 3(p-1)/p = (3-2p)/p < 0$. Then, taking $\delta = \max(r,\, 1)$, we obtain $I_2 \leq kr$. Finally, let $\delta = r$. Then the remaining terms in $I_1 + I_2$ to be estimated are the following:

$$r \int_{\substack{r < |X-Y| < 1 \\ Y \in \Omega}} |\, \omega(t,\, Y)\, |\, |\, X - Y\, |^{-3} dY \ (\text{when } r < 1),$$

$$\int_{\substack{1 < |X-Y| < r \\ Y \in \Omega}} |\, \omega(t,\, Y)\, |\, |\, X - Y\, |^{-2} dY \ (\text{when } r > 1).$$

The former term is estimated as in (5.3) by taking p, p' as $5/4 < p < 3/2$, $p' = s'$. The latter term is also treated as in (5.2) by taking q, q' as $3/2 < q < 2$, $q' = s''$. So we obtain the estimate (3.2) and complete the proof of Lemma.

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