64. On a Strict Decomposition of Additive Functionals for Symmetric Diffusion Processes

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1. Introduction. Let X be a locally compact separable metric space and m be a positive Radon measure on X with full support. For an m-symmetric Hunt process $\boldsymbol{M} = (X_t, P_x)$ on X with the associated Dirichlet form being regular, the following decomposition of additive functionals (AF's in abbreviation) has been known ([1], [3]):

(1) $$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]},\ P_x\text{-almost surely}$$

which holds for quasi every (q. e. in abbreviation) $x \in X$. Here u is a function in the Dirichlet space, $M_t^{[u]}$ is a martingale AF of finite energy, $N_t^{[u]}$ is a continuous AF of zero energy and 'for q.e. $x \in X$' means 'for every $x \in X$ outside a set of zero capacity'. $N_t^{[u]}$ is then of zero quadratic variation on each finite time interval P_m-a.s. but not necessarily of bounded variation. In this sense, (1) is beyond a semimartingale decomposition and it is a prototype of the so called Dirichlet process. However we can not tell in general where the exceptional set of zero capacity involved in the decomposition (1) is located. This ambiguity imposes a limitation on its applicability especially to the finite dimensional analysis.

If we assume the absolute continuity of the transition function $p_t(x, B)$ of the process $\boldsymbol{M}$:

(2) $$p_t(x, \cdot) \prec m,\ \forall t > 0,\ \forall x \in X,$$

then it is possible to refine the decomposition (1) by giving conditions on the function u so that the AF's on the right hand side of (1) make sense for every starting point $x \in X$ (namely, they are converted into AF's in the strict sense) and further (1) holds for every $x \in X$ as well. Some sufficient conditions for this are presented in the book [3]. In [2], the author shows that a necessary and sufficient condition for this is that the energy measure of u is smooth in the strict sense:

(3) $$\mu_{\langle u \rangle} \in S_1.$$

In this paper, we start with conditions (2) and (3) and investigate some basic properties of the corresponding AF $N_t^{[u]}$ in the strict sense. To simplify the presentation, we assume that the Dirichlet form is strongly local and $\boldsymbol{M}$ is a conservative diffusion process. We can then deal with functions belonging locally to the Dirichlet space.

2. A strict decomposition. We use those notions and notations in [3] concerning Dirichlet forms, diffusion processes and additive functionals (AF's in abbreviation). Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular Dirichlet form on $L^2(X ; m)$. We assume that there exists a conservative diffusion process

$\boldsymbol{M} = (\Omega, \mathscr{F}_t, X_t, P_x)$ on X associated with $\mathscr{E}$ whose transition function p_t satisfies the absolute continuity condition (2). We note that, although the existence of a diffusion process associated with the present Dirichlet form $\mathscr{E}$ follows automatically from general theorems in [3], the existence of such a process possessing the additional property (2) is highly non-trivial, and in many cases we have to work with other methods like PDE in the construction. See [4] for a prototype of such a construction.

We say that a function u is locally in $\mathscr{F}$ ($u \in \mathscr{F}_{loc}$ in notation) if for any relatively compact open set G there exists a function $w \in \mathscr{F}$ such that $u = w$ m-a.e. on G. Any $u \in \mathscr{F}$ admits a unique positive Radon measure $\mu_{\langle u \rangle}$ called the *energy measure* of u, which satisfies $\mathscr{E}(u, u) = \frac{1}{2}\mu_{\langle u \rangle}(X)$. Under the present locality assumption on $\mathscr{E}$, the energy measure can be also associated with $\mathscr{F}_{loc}$ uniquely.

A real valued function $A_t(\omega)$ of $t \geq 0$ and $\omega \in \Omega$ is called a *continuous AF* (CAF in abbreviation) *in the strict sense* if it is $\{\mathscr{F}_t\}$-adapted and the following properties in $t \geq 0$ hold P_x-a.s. $\forall x \in X : A_0(\omega) = 0$, $A_t(\omega)$ is continuous on $[0, \infty)$ and additive: $A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s\omega)$, $s, t \geq 0$. Two CAF's in the strict sense $A^{(1)}, A^{(2)}$ are regarded to be equivalent if they are *indistinguishable* in the sense that $A_t^{(1)} = A_t^{(2)}$ $\forall t > 0$ P_x-a.s. $\forall x \in X$.

A CAF in the strict sense is called *positive* (a PCAF in the strict sense in abbreviation) if it is non-negative $\forall t \geq 0$ P_x-a.s. $\forall x \in X$. The equivalence classes of all PCAF's in the strict sense is denoted by $\boldsymbol{A}_{c1}^{+}$.

We add the phrase 'in the strict sense' to an AF in order to distinguish it from the somewhat relaxed notion of an AF employed in [3] which admits an exceptional set of starting points $x \in X$ of zero capacity and fits more in the Dirichlet form setting. Under the present absolute continuity assumption (2) however, we can handle AF's in the strict sense equally in a systematic way.

(2) implies that the resolvent kernel $R_\alpha(x, E)$ of $\boldsymbol{M}$ admits a symmetric density $r_\alpha(x, y)$ with respect to m which is α-excessive in each variable ([3, Lemma 4.2.4]). For a positive Borel measure μ on X, its α-potential is defined by $R_\alpha\mu(x) = \int_X r_\alpha(x, y)\mu(dy)$, $x \in X$. Denote by S_{00} the family of positive Borel measures μ such that $\mu(X) < \infty$ and $\sup_{x \in X} R_\alpha\mu(x) < \infty$. An increasing sequence $\{E_l\}$ of finely open sets with $\cup_{l=1}^{\infty} E_l = X$ will be called an *exhaustive sequence* of finely open sets. A positive Borel measure μ is said to be *smooth in the strict sense* if there exists an exhaustive sequence of finely open sets $\{E_l\}$ such that $I_{E_l}\cdot\mu \in S_{00}$, $l = 1, 2, \cdots$.

Denote by S_1 the totality of smooth measures in the strict sense. S_1 is known to stand in one to one correspondence with the family $\boldsymbol{A}_{c1}^{+}$ of functionals by the Revuz correspondence ([3, Th. 5.1.7]). S_1 is contained in the family S of smooth measures introduced in [3, §2.2].

The next theorem is formulated and proven in [2, Theorem 2] without the present conservativeness assumption. We use the notations $\mathring{\mathscr{M}}_{loc}$ (resp.

$\mathcal{N}_{c,loc}$) from [3, §5.5] standing for the family of all martingale AF's locally of finite energy (resp. all continuous AF's locally of zero energy).

Theorem 2.1. *Suppose*

(4) u is finite valued, finely continuous and $u \in \mathcal{F}_{loc}$.

The condition (3) *is then necessary and sufficient for* u *to admit the decomposition*

(5) $u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \ \forall t \geq 0 \ P_x$-a. s., $\forall x \in X$,

with $M^{[u]}$, $N^{[u]}$ *possessing the following properties*:

(a) $M^{[u]} \in \mathcal{M}_{loc}$, $M^{[u]}$ *is a CAF in the strict sense admitting an* $A \in \boldsymbol{A}_{c1}^{+}$ *and an exhaustive sequence* $\{E_l\}$ *of finely open sets such that, for each* l,

$$E_x((M_{t\wedge\tau_l}^{[u]})^2) = E_x(A_{t\wedge\tau_l}) < \infty, \ E_x(M_{t\wedge\tau_l}^{[u]}) = 0 \ \forall x \in E_l.$$

Here τ_l *denotes the first leaving time from the set* E_l.

(b) $N^{[u]} \in \mathcal{N}_{c,loc}$, $N^{[u]}$ *is a CAF in the strict sense.*

In this case, the energy measure $\mu_{\langle u \rangle}$ *and* A *in* (a) *are related by the Revuz correspondence. Any exhaustive sequence of finely open sets associated with* $\mu_{\langle u \rangle}$ *works as a sequence appearing in* (a).

The above decomposition is unique up to the indistinguishability.

Theorem 2.1 (a) means that $M^{[u]}$ is a local martingale CAF (in the strict sense) with the quadratic variation being the PCAF in the strict sense corresponding to the energy measure $\mu_{\langle u \rangle}$.

3. Properties of $N^{[u]}$. In this section, we consider those functions u satisfying conditions (3) and (4) and investigate the locality, the support and the absolute variation of the corresponding CAF's $N^{[u]}$ in the strict sense, locally of zero energy, produced by Theorem 2.1.

Theorem 3.1. *Suppose that* u_1, u_2 *satisfy conditions* (3) *and* (4) *and that* $u_1 - u_2$ *is constant on a nearly Borel finely open set* G. *Then*

(6) $$M_{t\wedge\tau_G}^{[u_1]} = M_{t\wedge\tau_G}^{[u_2]}, \ N_{t\wedge\tau_G}^{[u_1]} = N_{t\wedge\tau_G}^{[u_2]}$$

up to indistinguishability. Here τ_G *denotes the first leaving time from* G.

Proof. By virtue of [3, Lemma 5.5.1], (6) holds $\forall t \geq 0 \ P_x$-a.s. for q.e. $x \in X$. This can be strengthened to " $\forall x \in X$" in the same way as in [2, Proof of uniqueness].

The *sprectrum* $\sigma(u)$ of a function $u \in \mathcal{F}_{Loc}$ is defined in [3] as the complement of the largest open set G such that $\mathcal{E}(u, v) = 0 \ \forall v = \mathcal{C}_G$, $\mathcal{C}$ being any special standard core of $\mathcal{E}$ and $\mathcal{C}_G = \{v \in \mathcal{C} : \mathrm{supp}[u] \subset G\}$. The α-spectrum $\sigma_a(u)$ is defined by replacing $\mathcal{E}$ with $\mathcal{E}_\alpha$ in the above.

For a closed set F, we write $\{t > 0 : X_t(\omega) \in X - F\} = \cup_\eta I_\eta(\omega)$, $\{I_\mu\}$ being countable number of disjoint open intervals, which can be enumerated in a way that ends points are measurable. We call $\{I_\eta\}$ excursions of the sample path X_t out of F.

Theorem 3.2. *Suppose that* $u \in \mathcal{F}$ *and* u *satisfies conditions* (3) *and* (4). *Let* $\{I_\eta\}$ *be excursions out of the spectrum* $\sigma(u)$ *of* u. *Then*

(7) $P_x(N_t^{[u]}$ is constant on $I_\eta \, \forall \eta) = 1, \ \forall x \in X$.

Let $\{I_\eta\}$ *be excursions out of the* α*-spectrum* $\sigma_\alpha(u)$ *of* u. *Then*

(8) $$P_x\left(N_t^{[u]} - \alpha \int_0^t u(X_s)ds \text{ is constant on } I_\eta, \ \forall \eta\right) = 1, \ \forall x \in X.$$

Proof. A weaker version of the above theorem with '$\forall x \in X$' being replaced by 'for q.e. $x \in X$' is implied in the proof of [3, Theorem 5.4.1], where the Beurling-Deny theorem on the spectral synthesis is invoked. Then we can utilize the condition (2) to get for any $x \in X$ and $\varepsilon > 0$

$$\begin{aligned} &P_x(N_t^{[u]} \text{ is not constant on } [\varepsilon, \infty) \cap I_\eta \ \exists \eta) \\ &= P_x(N_t^{[u]}(\theta_\varepsilon \omega) \text{ is not constant on } I_\eta(\theta_\varepsilon \omega) \ \exists \eta) \\ &= E_x(P_{X_\varepsilon}(N_t^{[u]} \text{ is not constant on } I_\eta \ \exists \eta)) = 0 \end{aligned}$$

arriving at (7). The proof of (8) is the same.

Remark. On account of Theorem 3.1 and the derivation property of the energy measure ([3, §3.2]), We can prove that Theorem 3.2 extends to $u \in \mathscr{F}_{b,loc}$ provided that there exist an exhaustive sequence $\{G_k\}$ of relatively compact open sets and functions $\{\phi_k\}$ in $\mathscr{F} \cap C_0(X)$ such that ϕ_k equals 1 on G_k, vanishes outside G_{k+1} and $\mu_{\langle \phi_k \rangle} \in S_1$, $k = 1,2, \cdots$.

For a subset $B \subset X$, we let $\mathscr{F}_{b,B} = \{v \in \mathscr{F} \cap L^\infty : \tilde{v} = 0 \text{ q.e. on } X - B\}$.

Theorem 3.3. *The next two conditions* (3.a) *and* (3.b) *are equivalent for a function u satisfying* (3) *and* (4):

(3.a) *There exists a signed measure μ expressible as $\mu = \mu_1 - \mu_2$ for some μ_1, $\mu_2 \in S_1$ and*

$$\mathscr{E}(u, v) = \langle \mu, \tilde{v} \rangle \ \forall v \in \bigcup_k \mathscr{F}_{b,G_k}$$

where $\{G_k\}$ is an exhaustive sequence of finely open sets commonly associated with μ_1, μ_2.

(3.b) $N_t^{[u]}$ *is of bounded variation on each compact interval of* $[0, \infty)$ P_x-a.s. $\forall x \in X$.

In this case, $N^{[u]} = -A^{(1)} + A^{(2)}$ *for* $A^{(i)} \in \mathbf{A}_{c,1}^{+}$ *with Revuz measure* μ_i, $i = 1,2$.

Proof of (3.a)$\Rightarrow$(3.b). Let $A^{(i)}$, $i = 1,2$, be as in the last assertion above and set $A = A^{(1)} - A^{(2)}$. Then, we get from (3.a) and [3, Lemma 5.4.4]

$$P_x(N_t^{[u]} = -(I_{G_k} \cdot A)_t (= -A_t),\ t < \tau_{G_k}) = 1 \text{ q.e. } x \in X.$$

By letting $k \to \infty$, $P_x(N_t^{[u]} = -A_t \ \forall t \geq 0) = 1$ q.e. $x \in X$, and by virtue of (2)

$$P_x(N_{t+\varepsilon}^{[u]} - N_\varepsilon^{[u]} = -A_{t+\varepsilon} + A_\varepsilon \ \forall t \geq 0) = 1 \ \forall x \in X.$$

Since both $N^{[u]}$ and A are AF's in the strict sense, we may let $\varepsilon \downarrow 0$ to get

$$P_x(N_t^{[u]} = -A_t \ \forall t \geq 0) = 1 \ \forall x \in X.$$

Proof of (3.b)$\Rightarrow$(3.a). As in the proof of [3, Theorem 5.4.2], it suffices to use the following variant of [3, Lemma 5.4.3]:

Lemma 3.1. *For any $N \in \mathscr{N}_c$ and for any nearly Borel finely open set G,*

$$\lim_{t \downarrow 0} \frac{1}{t} E_{v \cdot m}(N_t \,;\, t > \tau_G) = 0, \ \forall v = R_\alpha^G f,\ f \in L_b^1(X \,;\, m).$$

Since the functional $N^{[u]}$ is strict version of that appearing in [3], we can restate [3, Theorem 5.5.4] as follows:

Theorem 3.4. *The following two conditions* (3.c) *and* (3.d) *are equivalent for a function u satisfying* (3) *and* (4):

(3.c) *For some smooth signed measure μ and a generalized compact nest $\{F_k\}$ associated with μ,*

$$\mathscr{E}(u, v) = \langle \mu, \tilde{v} \rangle \quad \forall v \in \bigcup_k \mathscr{F}_{b,F_k}.$$

(3.d) $N_t^{[u]}$ *is of bounded variation on each compact interval of* $[0, \infty)$ P_x*-a.s. for q.e.* $x \in X$.

In this case, the following is true:

(3.e) $N_t^{[u]}$ *is of bounded variation on each compact interval of* $(0, \infty)$ P_x*-a.s.* $\forall x \in X$.

The last assertion is due to the absolute continuity condition (2).

Corollary 3.1. *Let* u *be a function satisfying* (3) *and* (4). *Suppose there exists a signed Radon measure* μ *satisfying for a special standard core* $\mathscr{C}$ *of* $\mathscr{E}$

$$\mathscr{E}(u, v) = \langle \mu, v \rangle \quad \forall v \in \mathscr{C}.$$

(i) *If* μ *charges no set of zero capacity, then* (3.d) *and* (3.e) *hold.*

(ii) *If the total variation of* μ *is in* S_1, *then* (3.b) *holds.*

In fact, as [3, Corollary 5.5.1] we can see that (i) (resp. (ii)) implies (3.c) (resp. (3.a)).

References

[1] M. Fukushima: Dirichlet Forms and Markov Processes. Kodansha and North-Holland (1980).

[2] ——: On a decomposition of additive functionals in the strict sense for a symmetric Markov processes. Proc. International Conference on Dirichlet Forms and Stochastic Processes, Beijing, 1993 (eds. Z. Ma, M. Röckner, and J. Yan). Walter de Gruyter (to appear).

[3] M. Fukushima, Y. Oshima and M. Takeda: Dirichlet Forms and Symmetric Markov Processes. Walter de Gruyter (1994).

[4] M. Fukushima and M. Tomisaki: Reflecting diffusions on Lipschitz domains with cusps-anlytic construction and Skorohod representations-. Proc. Conference on Potential Theory and Differential Operators with Non-Negative Characteristic Forms Parma, 1994 (ed. M. Biroli). Kluwer (to appear).