59. The Explicit Formula for the Harish-Chandra C-function of SU(n, 1) for Arbitrary Irreducible Representations of K which Contain One Dimensional M-types

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§0. Introduction. The purpose of this note is to give an explicit expression of the Harish-Chandra C-function of $SU(n, 1)$ for the case when irreducible unitary representations of the maximal compact subgroup $K = S(U(n) \times U(1))$ contain one dimensional unitary representation of $M = S(U(n-1) \times U(1))$. In this paper we compute the matrix element of the Harish-Chandra C-function with respect to the M-highest weight vector of the above M-type. Our main result will give a complete determination of the composition series of the representations which are induced from one dimensional unitary representations of M and characters of the noncompact Cartan subalgebra. Moreover in order to prove the Paley-Wiener type theorem, we need the information on the positions of zeros and poles and their orders of the Harish-Chandra C-function for every K-types which contain certain M-type.

In order to prove our result, using analogous arguments in [3], we obtain the recursion formulae of the Harish-Chandra C-function.

§1. Notation and preliminaries. Let $G = SU(n, 1)$ $(n \geq 2)$ and $K = S(U(n) \times U(1))$. Then K is a maximal compact subgroup of G. Define the analytic subgroups A, N and $\bar{N}$ by

$$A = \left\{ \begin{pmatrix} \cosh t & & \sinh t \\ & I_{n-1} & \\ \sinh t & & \cosh t \end{pmatrix}; t \in \boldsymbol{R} \right\},$$

$$N = \left\{ \begin{pmatrix} 1 - \omega/2 & z^* & \omega/2 \\ -z & I_{n-1} & z \\ -\omega/2 & z^* & 1 + \omega/2 \end{pmatrix}; z \in \boldsymbol{C}^n, u \in \boldsymbol{R}, \omega = \sum_{i=1}^{n-1} |z_i|^2 + 2\sqrt{-1}u \right\},$$

$$\bar{N} = \left\{ \begin{pmatrix} 1 - \omega/2 & z^* & -\omega/2 \\ -z & I_{n-1} & -z \\ \omega/2 & -z^* & 1 + \omega/2 \end{pmatrix}; z \in \boldsymbol{C}^n, u \in \boldsymbol{R}, \omega = \sum_{i=1}^{n-1} |z_i|^2 + 2\sqrt{-1}u \right\},$$

where I_{n-1} denotes the unit matrix of order $n-1$ and the asterisk denotes the conjugate transpose. Let M be the centralizer of A in K and $\mathfrak{a}$ be the Lie algebra of A. Then they are given by

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$$M=\left\{\begin{pmatrix} u & & \\ & X & \\ & & u \end{pmatrix};\, X\in U(n-1),\ u\in \boldsymbol{C},\ u^2\det X=1\right\},$$

$$\mathfrak{a}=\{tH\ ;\ t\in \boldsymbol{R}\},\ \text{where } H=\begin{pmatrix} & & & & 1\\ & & & 0 & \\ & & \cdot & & \\ & 0 & & & \\ 1 & & & & \end{pmatrix}.$$

For any $g\in G$, let $g=\kappa(g)\exp H(g)n(g)$ be the Iwasawa decomposition of g, where $\kappa(g)\in K$, $H(g)\in\mathfrak{a}$, $n(g)\in N$. Put $w=\operatorname{diag}(-1,-1,1,\dots,1)\in K$. Then w is a representative of the nontrivial element of the Weyl group of G.

The complex dual space $\mathfrak{a}_{\boldsymbol{C}}^*$ of $\mathfrak{a}$ can be identified with $\boldsymbol{C}$ under the correspondence $\lambda\in\mathfrak{a}_{\boldsymbol{C}}^*\to\lambda(H)\in\boldsymbol{C}$. Let ρ denote the rho function of G. Then ρ is identified with n.

We denote by $\hat{K}$ the set of equivalence classes of irreducible unitary representations of K. Then for $\tau\in\hat{K}$, the Harish-Chandra C-function is defined as the following integral:

$$C_\tau(\lambda)=\int_{\bar{N}}\tau(\kappa(\bar{n}))^{-1}e^{-(\lambda+\rho)(H(\bar{n}))}d\bar{n},\ (\lambda\in\mathfrak{a}_{\boldsymbol{C}}^*).$$

§2. Determination of K-types. In this section we will determine the subset of $\hat{K}$ consisting of the elements which contain one dimensional M-type. As in [4], $\hat{K}$ and $\hat{M}$ are parametrized as follows:

$$\hat{K}=\left\{s=(s_1,\dots,s_n)\in\left(\frac{1}{n+1}\boldsymbol{Z}\right)^n;\ s_j-s_{j+1}\in\boldsymbol{Z}_{\geq 0}\ (j=1,\dots,n-1)\right\},$$

$$\hat{M}=\left\{t=(t_1,\dots,t_{n-1})\in\left(\frac{1}{n+1}\boldsymbol{Z}\right)^{n-1};\ t_j-t_{j+1}\in\boldsymbol{Z}_{\geq 0}\ (j=1,\dots,n-2)\right\}.$$

It is known that (cf. [4]), for $s\in\hat{K}$ and $t\in\hat{M}$, $[s:t]=0$ or 1 and $[s:t]\neq 0$ iff $s_j-t_j\in\boldsymbol{Z}_{\geq 0}$ and $t_j-s_{j+1}\in\boldsymbol{Z}_{\geq 0}$ $(j=1,\dots,n-1)$.

Let σ^m $(m\in\boldsymbol{Z})$ be the one dimensional unitary representation of M on $\boldsymbol{C}$, so that

$$\sigma^m\left(\begin{pmatrix} u & & \\ & X & \\ & & u \end{pmatrix}\right)z=u^m z,\ \left(\begin{pmatrix} u & & \\ & X & \\ & & u \end{pmatrix}\in M,\ z\in\boldsymbol{C}\right).$$

We denote by $V_{p,q}$ the set of harmonic polynomials in $z\in{}^t\boldsymbol{C}^n$ of bidegree (p,q). We define the action $\tau_{m,p,q}$ $(m\in\boldsymbol{Z})$ of K on $V_{p,q}$ by

$$\left(\tau_{m,p,q}\left(\begin{pmatrix} X & \\ & u \end{pmatrix}\right)\varphi\right)(z)=u^{q-p+m}\varphi(zX),\ \left(\begin{pmatrix} X & \\ & u \end{pmatrix}\in K,\ X\in U(n),\ u\in\boldsymbol{C},\ \varphi\in V_{p,q}\right).$$

Under Kraljević's parameter, these representations can be written as follows:

$$\sigma^m=\left(-\frac{m}{n+1},\dots,-\frac{m}{n+1}\right),$$

$$\tau_{m,p,q}=\left(p-\frac{m}{n+1},-\frac{m}{n+1},\dots,-\frac{m}{n+1},-q-\frac{m}{n+1}\right).$$

Thus we have the following proposition.

Proposition 1. *Let τ be an arbitrary K-type which contains the M-type σ^m. Then there exist p, $q\in \boldsymbol{Z}_{\geq 0}$ such that τ is equivalent with $\tau_{m,p,q}$.*

Put $\varphi_{p,q}(z)=z_1^p\bar{z}_1^q F(-p,-q;n-1;-(|z_2|^2+\cdots+|z_n|^2)/|z_1|^2)$, where F denotes the standard hypergeometric function (cf. [3]). Then $\varphi_{p,q}\in V_{p,q}$. A simple calculation shows that

$$\left(\tau_{m,p,q}\left(\begin{pmatrix}u&&\\&X&\\&&u\end{pmatrix}\right)\varphi_{p,q}\right)(z)=u^m\varphi_{p,q}(z),\ (\begin{pmatrix}u&&\\&X&\\&&u\end{pmatrix}\in M).$$

§3. Intertwining operators and the C-functions. In this section we will give an explicit expression of the Harish-Chandra C-function. To do this, we shall first find the recursion formulae for the Harish-Chandra C-function. We use here the arguments in [6, p. 218-277]. Let $(\tau_{m,\lambda}, H^{m,\lambda})$ denote the principal series representation of G induced from the representation $\sigma^m\otimes\lambda\otimes 1$ of MAN and let $A(w,m,\lambda):H^{m,\lambda}\to H^{m,-\lambda}$ denote the standard intertwining operator. Define the linear mapping P of $V_{p,q}$ into $\boldsymbol{C}$ by $P(\varphi)=\varphi((1,0,\dots,0))$. Then P satisfies $P\tau_{m,p,q}(b)=\sigma^m(b)P$ for any $b\in M$ and $P(\varphi_{p,q})=1$. For $g\in G$, put $\tilde{\varphi}_{p,q,\lambda}(g)=e^{-(\lambda+\rho)H(g)}P(\tau_{m,p,q}(\kappa(g))^{-1}\varphi_{p,q})$. Then it is clear that $\tilde{\varphi}_{p,q,\lambda}\in H^{m,\lambda}$. By Schur's lemma, there exists a constant $a_{p,q}(\lambda)$ such that $A(w,m,\lambda)(\tilde{\varphi}_{p,q,\lambda})=a_{p,q}(\lambda)\tilde{\varphi}_{p,q,-\lambda}$. Noting that $[\tau_{m,p,q}:\sigma^m]=1$ and $\tau_{m,p,q}(w)\varphi_{p,q}=(-1)^{p+q}\varphi_{p,q}$, we have from [6, p. 270] that

$$a_{p,q}(\lambda)=(-1)^{p+q}C_{\tau_{m,p,q}}(\sigma^m:\lambda),$$

where $C_{\tau_{m,p,q}}(\sigma^m:\lambda)$ denote the matrix element of $C_{\tau_{m,p,q}}(\lambda)$ with respect to the element $\varphi_{p,q}$ which corresponds to the M-highest weight vector 1 of σ^m under P.

A straightforward calculation shows that (cf. [2], [3])

$$\begin{aligned}\pi_{m,\lambda}(H)\tilde{\varphi}_{p,q,\lambda}=\frac{1}{2(p+q+n-1)}\{&(p+n-1)(\lambda+n-m+2p)\tilde{\varphi}_{p+1,q,\lambda}\\&+p(\lambda-n-m-2(p-1))\tilde{\varphi}_{p-1,q,\lambda}\\&+(p+n-1)(\lambda+n+m+2p)\tilde{\varphi}_{p,q+1,\lambda}\\&+q(\lambda-n+m+2(q-1))\tilde{\varphi}_{p,q-1,\lambda}\}.\end{aligned}$$

The intertwining relationship between $\pi_{m,\lambda}$ and $\pi_{m,-\lambda}$ implies that $A(w,m,\lambda)\pi_{m,\lambda}(H)\tilde{\varphi}_{p,q,\lambda}=\pi_{m,-\lambda}(H)A(w,m,\lambda)\tilde{\varphi}_{p,q,\lambda}$. Therefore we obtain the following recursion formulae:

$$(\lambda+n-m+2p)a_{p+1,q}(\lambda)=(-\lambda+n-m+2p)a_{p,q}(\lambda),$$
$$(\lambda+n+m+2q)a_{p,q+1}(\lambda)=(-\lambda+n+m+2q)a_{p,q}(\lambda).$$

Thus we have

$$a_{p,q}(\lambda)=(-1)^{p+q}\prod_{j=0}^{p-1}\frac{\lambda-n+m-2j}{\lambda+n-m+2j}\prod_{j=0}^{q-1}\frac{\lambda-n-m-2j}{\lambda+n+m+2j}a_{0,0}(\lambda).$$

Here $a_{0,0}(\lambda)$ is the Harish-Chandra C-function for one dimensional K-type and it is computed as follows (cf. [5]):

$$a_{0,0}(\lambda) = \frac{(n-1)!2^{-\lambda+n}\Gamma(\lambda)}{\Gamma\left(\frac{\lambda+n+m}{2}\right)\Gamma\left(\frac{\lambda+n-m}{2}\right)}.$$

Therefore we have the following theorem.

Theorem 2. *We have the following expression*:

$$C_{\tau_{m,p,q}}(\sigma^m:\lambda) = \frac{(n-1)!2^{-\lambda+n-p-q}\Gamma(\lambda)\prod_{j=0}^{p-1}(\lambda-n+m-2j)\prod_{j=0}^{q-1}(\lambda-n-m-2j)}{\Gamma\left(\frac{\lambda+n-m+2p}{2}\right)\Gamma\left(\frac{\lambda+n+m+2q}{2}\right)}.$$

Remark. If $n=2$ then our theorem gives the explicit expression of the Harish-Chandra C-function of $SU(2,1)$ for all K-types (cf. [6]). If $p=q=1$ and $m=0$ then it coincides with the case of adjoint representation (cf. [1]).

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