57. *On Fractional Powers of a Class of Elliptic Differential Operators with Feedback Boundary Conditions. II*

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§1. Introduction. In the study of boundary control systems, fractional powers of elliptic differential operators are of special importance. They often appear in optimal control and stabilization problems, and play a central role there, We consider in this paper a system of differential operators $(\mathscr{L}, \tau)$ in a bounded domain Ω of $\mathbb{R}^m$ with the boundary Γ which consists of a finite number of smooth components of $(m-1)$-dimension. Actually, let $\mathscr{L}$ denote a uniformly elliptic differential operator of order 2 in Ω defined by

$$\mathscr{L}u = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

where $a_{ij}(x) = a_{ji}(x)$ for $1 \leq i, j \leq m$ and $x \in \bar{\Omega}$. Associated with $\mathscr{L}$ is a boundary operator τ of the Neumann or Robin type given by

$$\tau u = \frac{\partial u}{\partial \nu} + \sigma(\xi)u = \sum_{i,j=1}^{m} a_{ij}(\xi)\nu_i(\xi)\frac{\partial u}{\partial x_j} + \sigma(\xi)u,$$

where $(\nu_1(\xi), \dots, \nu_m(\xi))$ denotes the unit outer normal at $\xi \in \Gamma$. Necessary regularity on $\bar{\Omega}$ and on Γ of coefficients of $\mathscr{L}$ and τ is assumed tacitly. Moreover $\sigma(\xi)$ is assumed to have a suitable smooth extension to $\bar{\Omega}$. Let us define the linear operators L and M in $L^2(\Omega)$ by

$$Lu = \mathscr{L}u, \quad u \in \mathscr{D}(L) = \{u \in H^2(\Omega)\,;\, \tau u = 0 \text{ on } \Gamma\}$$

and

$$Mu = \mathscr{L}u, \quad u \in \mathscr{D}(M) = \left\{u \in H^2(\Omega)\,;\, \tau u = \sum_{k=1}^{p} \langle u, w_k \rangle_\Gamma h_k \text{ on } \Gamma\right\},$$

respectively. Here, $w_k \in L^2(\Gamma)$ stand for weight functions of observations distributed on Γ; h_k the actuators belonging to $H^{1/2}(\Gamma)$; $\langle \cdot, \cdot \rangle_\Gamma$ the inner product in $L^2(\Gamma)$; and p a positive integer depending on the control problems under consideration. Thus the boundary condition for M may be described as a feedback type. The operator M is not a standard type in the sense that the boundary condition is composed of terms of local nature and those of global nature. All norms hereafter will be $L^2(\Omega)$-or $\mathscr{L}(L^2(\Omega))$-norms unless otherwise indicated. As is well known [7], there is a sector $\overline{\Sigma}_{-\alpha} = \overline{\Sigma} - \alpha$, $\alpha > 0$, such that $\overline{\Sigma}_{-\alpha}$ is contained in the resolvent set $\rho(L)$, where $\overline{\Sigma} = \{\lambda\,;\, \theta \leq |\arg \lambda| \leq \pi\}$, $0 < \theta < \pi/2$, and the upper bar means the closure of a set. Choose a positive constant $c(> \alpha)$, and let $L_c = L + c$. Then fractional powers of the operator L_c are well defined. As is well known [2], we have the characterization of L_c^ω

(1) $\mathscr{D}(L_c^{\omega}) = H^{2\omega}(\Omega), \quad 0 \leq \omega < \dfrac{3}{4}$;

(2) $\mathscr{D}(L_c^{3/4}) = \{u \in H^{3/2}(\Omega)\ ;\ \int_{\Omega} \zeta(x)^{-1} |(\tau_{\Omega} u)(x)|^2 dx < \infty\}$; and

(3) $\mathscr{D}(L_c^{\omega}) = H_{\tau}^{2\omega}(\Omega) = \{u \in H^{2\omega}(\Omega)\ ;\ \tau u = 0 \text{ on } \Gamma\}, \quad \dfrac{3}{4} < \omega \leq 1$

with equivalence of the graph norms for the left sides and the Sobolev norms for the right sides. Here, $\zeta(x)$ denotes the distance from $x \in \Omega$ to $\partial\Omega$, and $\tau_{\Omega} = \partial/\partial\zeta + \sigma(x)$.

The purpose of the paper is to derive similar properties of the operator M corresponding to (1)-(3): They may be regarded as a considerable generalization of the result in our previous paper [6], in which only weight functions w_k distributed over Ω were considered and so the inner product in $L^2(\Omega)$: $\langle u, w_k \rangle$ was used instead of $\langle u, w_k \rangle_{\Gamma}$ in the present case, and the power ω is limited to less than $3/4$. In generalizing the result to the present case, however, some difficulties arise regarding m-accretiveness of the appearing operator and the limitation of the power ω. We adopt here a new approach to overcome the difficulties.

The sesquilinear form associated with M is given by

$$B[u, \varphi] = \sum_{i,j=1}^{m} \left\langle a_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial \varphi}{\partial x_i} \right\rangle + \sum_{i=1}^{m} \left\langle b_i \frac{\partial u}{\partial x_i}, \varphi \right\rangle + \langle cu, \varphi \rangle$$
$$+ \langle \sigma u, \varphi \rangle_{\Gamma} - \sum_{k=1}^{p} \langle u, w_k \rangle_{\Gamma} \langle h_k, \varphi \rangle_{\Gamma}.$$

Making use of the form $B[u, \varphi]$ and making necessary modifications of standard arguments for the elliptic boundary value problem [7]: $(\lambda - \mathscr{L})u = f \in L^2(\Omega)$, $\tau u = g \in H^{1/2}(\Gamma)$, we have the following

Proposition 1.1. *There is a* $\beta(> \alpha)$ *such that* $\overline{\Sigma}_{-\beta} = \overline{\Sigma} - \beta$ *is contained in* $\rho(M)$, *and that the following estimate holds*:

$$\| (\lambda - M)^{-1} \| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \overline{\Sigma}_{-\beta}.$$

Thus, $-M$ *generates an analytic semigroup* $\exp(-tM)$, $t > 0$.

§2. Main result. Our main result corresponds to the relations (1)-(3), and is stated as follows:

Theorem 2.1. *The domain of the fractional powers* M_c^{ω}, $0 \leq \omega \leq 1$, *is characterized as follows*:*)

(4) $\mathscr{D}(M_c^{\omega}) = H^{2\omega}(\Omega), \quad 0 \leq \omega < \dfrac{3}{4}$;

(5) $\mathscr{D}(M_c^{3/4}) = \left\{u \in H^{3/2}(\Omega)\ ;\ \int_{\Omega} \zeta(x)^{-1} \left| \tau_{\Omega} u - \sum_{k=1}^{p} \langle u, w_k \rangle_{\Gamma} \tau_{\Omega} R h_k \right|^2 dx < \infty \right\}$; and

*) The author would like to announce that a characterization of the domain of the fractional powers M_c^{ω}, $0 \leq \omega \leq 1$, corresponding to Theorem 2.1 has been recently obtained when the boundary operator τ is replaced by the operator of the Dirichlet type. This result will be reported in the author's forthcoming paper.

$$(6)\quad \mathscr{D}(M_c^{\omega}) = H_{\tau,f}^{2\omega} = \left\{u \in H^{2\omega}(\Omega)\ ;\ \tau u = \sum_{k=1}^{p} \langle u, w_k\rangle_\Gamma h_k \text{ on } \Gamma\right\}, \qquad \frac{3}{4} < \omega \leq 1,$$

where $R \in \mathscr{L}(H^{1/2}(\Gamma)\ ;\ H^2(\Omega))$ *is a (not unique) prolongation operator such that*

$$Rh\Big|_\Gamma = 0, \quad \text{and} \quad \frac{\partial}{\partial\nu} Rh\Big|_\Gamma = h, \quad h \in H^{1/2}(\Gamma).$$

Outline of the proof. Let us consider the following differential equation in $L^2(\Omega)$:

$$(7)\qquad \frac{du}{dt} + Mu = 0, \quad u(0) = u_0 \in L^2(\Omega).$$

Owing to Proposition 1.1, the problem (7) generates an analytic semigroup $\exp(-tM)$, $t > 0$ [4], and a unique solution is given by $u(t) = \exp(-tM)\, u_0$. For any given $h \in H^{1/2}(\Gamma)$, the boundary value problem; $(c + \mathscr{L})u = 0$ in Ω, and $\tau u = h$ on Γ admits a unique solution $u \in H^2(\Omega)$, which is denoted by Nh. The operator N belongs to $\mathscr{L}(H^{1/2}(\Gamma)\ ;\ H^2(\Omega))$. For any ϑ, $1/4 < \vartheta < 3/4$, set $v(t) = L_c^{-\vartheta} u(t)$. Then we see that $v(t)$ belongs to $\mathscr{D}(L)$ by (1) and that $v(t)$ satisfies the following differential equation in $L^2(\Omega)$;

$$\frac{dv}{dt} + (L - F)v = 0, \quad v(0) = v_0 = L_c^{-\vartheta} u_0,$$

where

$$Fv = \sum_{k=1}^{p} \langle L_c^{\vartheta} v, w_k\rangle_\Gamma L_c^{1-\vartheta} Nh_k, \quad \mathscr{D}(F) \supset \mathscr{D}(L).$$

Here we note that, if u is in $\mathscr{D}(M)$, then the function $u - \sum_{k=1}^{p} \langle u, w_k\rangle_\Gamma Nh_k$ is in $\mathscr{D}(L)$.

Lemma 2.2. *The operator* $L - F$ *has a compact resolvent. There is a* $\gamma > 0$ *such that* $\overline{\Sigma}_{-\gamma}$ *is contained in* $\rho(L - F)$, *and that*

$$\|(\lambda - L + F)^{-1}\| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \overline{\Sigma}_{-\gamma}.$$

It is not difficult to show that

$$(8)\qquad (\lambda - M)^{-1} = L_c^{\vartheta}(\lambda - L + F)^{-1} L_c^{-\vartheta}$$

for $\mathrm{Re}\,\lambda < -\gamma$. The right-hand side of eqn. (8) is analytic in $\lambda \in \rho(L - F)$. Thus, $(\lambda - M)^{-1}$ has an extension to an operator analytic in $\lambda \in \rho(L - F)$. The extension is, however, nothing but the resolvent of M [1]. This shows that $\rho(L - F)$ is contained in $\rho(M)$, and that eqn. (8) holds for $\lambda \in \rho(L - F)$ (more is true. See Corollary 2.4 below).

Choose a larger $c(> \max(\beta, \gamma))$, if necessary, and consider the fractional powers of $M_c = M + c$ and $L_c - F$. According to (8) valid for $\lambda \in \rho(L - F)$, we can show that

$$(9)\qquad M_c^{-\vartheta} = L_c^{\vartheta}(L_c - F)^{-\vartheta} L_c^{-\vartheta}.$$

We prove Lemma 2.3 below. Since the m-accretiveness of the operator $L_c - F$ is not expected, we take another approach different from the one in [6]: Essentially due to [4, Lemma 7.3], we see that, for $0 \leq \alpha < \beta$, the relations

$$\mathscr{D}((L_c - F)^\beta) \subset \mathscr{D}(L_c^\alpha), \text{ and } \mathscr{D}(L_c^\beta) \subset \mathscr{D}((L_c - F)^\alpha)$$

hold algebraically and topologically. Note that

$$(L_c - F)^{-\omega} - L_c^{-\omega} = \frac{1}{2\pi i}\int_C \lambda^{-\omega}(\lambda - L_c + F)^{-1} F(\lambda - L_c)^{-1} d\lambda$$
$$= \frac{1}{2\pi i}\int_C \lambda^{-\omega}(\lambda - L_c)^{-1} F(\lambda - L_c + F)^{-1} d\lambda, \quad 0 \le \omega \le 1,$$

where C denotes the contour; a suitable translation to the right of $\partial\Sigma$ in the right half-plane, oriented according to increasing $\mathrm{Im}\,\lambda$. Combining these relations, we have

Lemma 2.3. *The equivalence relation* $\mathscr{D}((L_c - F)^\omega) = \mathscr{D}(L_c^\omega)$, $0 \le \omega < 3/4 + \vartheta$ *holds algebraically and topologically.*

According to Lemma 2.3, we see that

$$L_c^\vartheta(L_c - F)^\vartheta L_c^{-2\vartheta} = L_c^\vartheta(L_c - F)^{-\vartheta} (L_c - F)^{2\vartheta} L_c^{-2\vartheta} \in \mathscr{L}(L^2(\Omega)),$$

since $2\vartheta < 3/4 + \vartheta$. Thus, the relation (9) implies that, for any $u \in \mathscr{D}(L_c^\vartheta)$,

$$M_c^{-\vartheta}(L_c^\vartheta(L_c - F)^\vartheta L_c^{-\vartheta} u) = u, \text{ or } M_c^\vartheta u = L_c^\vartheta(L_c - F)^\vartheta L_c^{-\vartheta} u,$$

which shows that $\mathscr{D}(L_c^\vartheta)$ is contained in $\mathscr{D}(M_c^\vartheta)$, and that

$$\| M_c^\vartheta u \| \le \text{const} \| L_c^\vartheta u \|, \quad u \in \mathscr{D}(L_c^\vartheta).$$

As to the converse relation, set $v = M_c^\vartheta u$ for $u \in \mathscr{D}(M_c^\vartheta)$. Then,

$$u = L_c^\vartheta(L_c - F)^{-\vartheta} L_c^{-\vartheta} v = L_c^{-\vartheta} L_c^{2\vartheta}(L_c - F)^{-2\vartheta} (L_c - F)^\vartheta L_c^{-\vartheta} v \in \mathscr{D}(L_c^\vartheta),$$

which shows that $\mathscr{D}(M_c^\vartheta)$ is contained in $\mathscr{D}(L_c^\vartheta)$, and that

$$\| L_c^\vartheta u \| \le \text{const} \| M_c^\vartheta u \|, \quad u \in \mathscr{D}(M_c^\vartheta).$$

Therefore, we have shown that $\mathscr{D}(M_c^\vartheta) = \mathscr{D}(L_c^\vartheta)$ with equivalent graph norms for any ϑ, $1/4 < \vartheta < 3/4$. For a fixed ϑ, $1/4 < \vartheta < 3/4$, a generalization of the Heinz inequality [3] is applied to M_c^ϑ and L_c^ϑ to derive that

$$\mathscr{D}(M_c^\omega) = \mathscr{D}((M_c^\vartheta)^{\omega/\vartheta}) = \mathscr{D}((L_c^\vartheta)^{\omega/\vartheta}) = \mathscr{D}(L_c^\omega), \quad 0 \le \omega \le \vartheta$$

with equivalent graph norms, which proves (4) of the theorem.

The proof of (5) and (6) is carried out as follows: As has been just proved, we note that $\mathscr{D}(M_c^{1/2}) = H^1(\Omega)$. Let us define an operator T formally by

$$v = Tu = u - \sum_{k=1}^{p} \langle u, w_k\rangle_\Gamma Rh_k. \tag{10}$$

It is not difficult to see that T is injective (namely, its formal inverse T^{-1} exists), and that

$$T \in \mathscr{L}(\mathscr{D}(M)\,;\mathscr{D}(L)) \cap \mathscr{L}(\mathscr{D}(M_c^{1/2})\,;\mathscr{D}(L_c^{1/2})), \text{ and}$$
$$T^{-1} \in \mathscr{L}(\mathscr{D}(L)\,;\mathscr{D}(M)) \cap \mathscr{L}(\mathscr{D}(L_c^{1/2})\,;\mathscr{D}(M_c^{1/2})).$$

Note that both M_c and L_c are m-accretive. Then, by the interpolation theory

$$T \in \mathscr{L}([\mathscr{D}(M), \mathscr{D}(M_c^{1/2})]_\theta\,; [\mathscr{D}(L), \mathscr{D}(L_c^{1/2})]_\theta) \tag{11}$$
$$= \mathscr{L}(\mathscr{D}(M_c^{1-\theta/2})\,;\mathscr{D}(L_c^{1-\theta/2})), \quad \text{and}$$

$$T^{-1} \in \mathscr{L}(\mathscr{D}(L_c^{1-\theta/2})\,;\mathscr{D}(M_c^{1-\theta/2})), \quad 0 \le \theta \le 1, \tag{12}$$

(see, for example [5, Theorem 6.1]). Thus we see that, for any $u \in \mathscr{D}(M_c^\omega)$, $3/4 < \omega \le 1$, $v = Tu$ belongs to $H_\tau^{2\omega}(\Omega)$ by (3), and that $\tau u = \sum_{k=1}^{p} \langle u, w_k\rangle_\Gamma h_k$. Therefore u belongs to $H_{\tau,f}^{2\omega}(\Omega)$. Conversely, for any $u \in H_{\tau,f}^{2\omega}(\Omega)$, $v = Tu$ belongs to $H_\tau^{2\omega}(\Omega) = \mathscr{D}(L_c^\omega)$. Thus u belongs to $\mathscr{D}(M_c^\omega)$ according to the relation (12), which proves (6) of the theorem. The relation (5) is similarly proved by means of the operator T. **Q.E.D.**

We close the paper with the following

Corollary 2.4. *For any $\omega \in \mathbb{R}^1$, M_c^{ω} is similar to $(L_c - F)^{\omega}$ in the sense that*

(13) $$M_c^{\omega} = L_c^{\vartheta}(L_c - F)^{\omega} L_c^{-\vartheta}, \text{ and } \rho(M_c^{\omega}) = \rho((L_c - F)^{\omega}).$$

Details of the proofs and related control theoretic results will appear elsewhere.

References

[1] N. Dunford and J. T. Schwartz: Linear Operators. vol. 2, Interscience, New York (1963).
[2] D. Fujiwara: Concrete characterization of the domain of fractional powers of some elliptic differential operators of the second order. Proc. Japan Acad., **43**, 82–86 (1967).
[3] T. Kato: A generalization of the Heinz inequality. ibid., **37**, 305–308 (1961).
[4] S. G. Krein: Linear Differential Equations in Banach Space. Amer. Math. Soc. Providence (1971).
[5] J. L. Lions and E. Magenes: Non-Homogeneous Boundary Value Problems and Applications. vol. 1, Springer-Verlag, New York (1972).
[6] T. Nambu: On fractional powers of a class of elliptic differential operators with feedback boundary conditions. Proc. Japan Acad., **66A**, 287–290 (1990).
[7] L. Nirenberg: Remarks on strongly elliptic partial differential equations. Comm. Pure Appl. Math., **8**, 649–675 (1955).