48. A Certain Formal Power Series Attached to Local Densities of Quadratic Forms. II

By Hidenori KATSURADA

Muroran Institute of Technology

(Communicated by Shokichi IYANAGA, M. J. A., June 7, 1994)

In this note, we announce some further results we have obtained as continuation of our previous papers [3], [4] on the formal power series attached to local densities of quadratic forms over the p-adic field. The power series we are treating now are not the same as those considered in [3], [4]. But the main results of [3], [4] can be deduced from the results of the present paper as explained in Remark 1 below. Concerning the matrices S and T of the quadratic forms, we suppose now only that S is even integral unimodular and T is diagonal with diagonal components satisfying certain conditions on ord_p. (Notations S, T and others are explained below.) This is a special case, but important special case of our present problem. Details will appear elsewhere.

Let p be an arbitrary prime number. For non-degenerate symmetric matrices S and T of degree m and n, respectively, with entries in the ring $\boldsymbol{Z}_p$ of p-adic integers, we define the local density $\alpha_p(T, S)$ and the primitive local density $\beta_p(T, S)$ by

$$\alpha_p(T, S) = \lim_{e\to\infty} p^{(-mn+n(n+1)/2)e} \# \mathscr{A}_e(T, S),$$

and

$$\beta_p(T, S) = \lim_{e\to\infty} p^{(-mn+n(n+1)/2)e} \# \mathscr{B}_e(T, S),$$

respectively, where

$$\mathscr{A}_e(T, S) = \{\bar{X} \in M_{m,n}(\boldsymbol{Z}_p)/p^e M_{m,n}(\boldsymbol{Z}_p) \ ; {}^tXSX \equiv T \bmod p^e\},$$

and

$$\mathscr{B}_e(T, S) = \{\bar{X} \in \mathscr{A}_e(T, S) \ ; X \text{ is primitive}\}.$$

Let A be an even integral unimodular matrix with entries in $\boldsymbol{Z}_p$. That is, A is a symmetric unimodular matrix with entries in $\boldsymbol{Z}_p$ whose diagonal components belong to $2\boldsymbol{Z}_p$. Then there exists a non-negative integer r such that A is equivalent, over $\boldsymbol{Z}_p$, to

$$\mathrm{diag}(\overbrace{H,\ldots,H}^{r}, U),$$

where we write $\mathrm{diag}(X, Y) = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ for two square matrices X, Y, and $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and U is an anisotropic even integral unimodular matrix of degree not greater than 2. Here we make the convention that $\mathrm{diag}(H,\ldots,H, U) = U$ or $= \mathrm{diag}(H,\ldots,H)$ according as $r = 0$ or $\deg U = 0$. We note that r is the Witt index of A, which will be denoted by $r(A)$. Then we define a matrix $A^{(k)}$ by

$$A^{(k)} = \mathrm{diag}(\overbrace{H,\dots,H}^{r-k}, U).$$

This $A^{(k)}$ is uniquely determined only by A up to equivalence over $\boldsymbol{Z}_p$. Further for each integers i, j, k such that $1 \le k \le i$, put

$$\gamma(i, j, k) = (-1)^k \sum_{0 \le i_1 < .. < i_k \le i-1} p^{(i-i_1)(j+i_1)} \cdots p^{(i-i_k)(j+i_k)}.$$

Then our main result is

Theorem 1. *Let the notation and the assumptions be as above. Put $e_p = 1$ or 0 according as $p = 2$ or not, and $m_0 = \min(t-1, r(A))$. Let $B_1 = \mathrm{diag}(b_1, \dots, b_t)$ and $B_2 = \mathrm{diag}(b_{t+1}, \dots, b_n)$ with $b_i \in \boldsymbol{Z}_p \setminus \{0\}$, and e be an integer such that $e \ge \mathrm{ord}_p(b_j)/2 - \mathrm{ord}_p(b_k)/2 + m_0 + 1 + e_p$ for $j = t+1, \dots, n$, $k = 1, \dots, t$. Then we have*

$$\alpha_p(\mathrm{diag}(p^{2e}B_1, B_2), A) = -\sum_{i=1}^{m_0} \gamma(t, -m+n+1, i)\alpha_p(\mathrm{diag}(p^{2(e-i)}B_1, B_2), A)$$
$$+ \Big(\prod_{i=0}^{m_0} \frac{1-p^{(t-i)(-m+n+i+1)}}{1-p^{-m+n+i+1}}\Big)\beta_p(O_{m_0+1}, A)\alpha_p(B_2, A^{(m_0+1)}),$$

where O_{m_0+1} is the zero matrix of m_0+1. Here we make the convention that the second term on the righ-hand side of the above equation is 0 if $r(A) = m_0$, and that we have $\alpha_p(B_2, A^{(m_0+1)}) = 1$ if $n = t$.

Now for non-degenerate symmetric matrices $B_1, \dots, B_s$, and A with entries in $\boldsymbol{Z}_p$ we define a define a formal power series $R((B_1, \dots, B_s), A; x_1, \dots, x_s)$ by

$$R((B_1,\dots,B_s), A; x_1,\dots,x_s)$$
$$= \sum_{e_1 \ge \dots \ge e_s \ge 0} \alpha_p(\mathrm{diag}(p^{e_1}B_1, \dots, p^{e_s}B_s), A)x_1^{e_1}\dots x_s^{e_s}.$$

Then by Theorem 1 we obtain easily

Theorem 2. *Let A be as in Theorem 1, and $B_i = \mathrm{diag}(b_{n_1+..+n_{i-1}+1}, \dots, b_{n_1+\dots+n_i})$ $(i = 1, \dots, s)$ with $b_j \in \boldsymbol{Z}_p \setminus \{0\}$. For $k = 1, \dots, s$ put $m_k = \min(n_1 + \dots + n_k - 1, r(A))$. Assume that $[\mathrm{ord}_p(b_j)/2] \ge [\mathrm{ord}_p(b_{j'})/2]$ for any $j' \ge n_1 + 1$ and $j \le n_1$. Then we have*

$$\prod_{i=0}^{m_1} (1 - p^{(n_1-i)(-m+n+i+1)}x_1^2)R((B_1, B_2, \dots, B_s), A; x_1, \dots, x_s)$$
$$= \sum_{i=0}^{m_1+e_p} x_1^{2i} \sum_{j=0}^{i} \gamma(n_1, -m+n+1, i-j)R((\mathrm{diag}(p^{2j}B_1, B_2), B_3, \dots, B_s), A; x_1x_2, x_3, \dots, x_s)$$
$$= \sum_{i=0}^{m_1+e_p} x_1^{2i+1} \sum_{j=0}^{i} \gamma(n_1, -m+n+1, i-j)R((\mathrm{diag}(p^{2j+1}B_1, B_2), B_3, \dots, B_s), A; x_1x_2, x_3, \dots, x_s)$$
$$+ \Big(\prod_{i=0}^{m_1} \frac{1-p^{(n_1-i)(-m+n+i+1)}}{1-p^{-m+n+i+1}}\Big)\beta_p(O_{m_1+1}, A)\frac{x_1^{2m_1+2+2e_p}}{1-x_1} R((B_2, \dots, B_s), A^{(m_1+1)}; x_1x_2, x_3, \dots, x_s).$$

Here we make the convention that $R((\mathrm{diag}(p^kB_1, B_2), B_3, \dots, B_s), A; x_1x_2, x_3, \dots, x_s) = \alpha_p(p^kB_1, A)$ and $R((B_2, \dots, B_s), A^{(m_1+1)}; x_1x_2, \dots, x_s)) = 1$ if $s = 1$.

Using Theorem 2, we can prove the following theorem by induction on s.

Theorem 3. *Assume that* $[\mathrm{ord}_p(b_j)/2] \geq [\mathrm{ord}_p(b_{j'})/2]$ *for any* $j' \geq n_1 + \ldots + n_i + 1$ *and* $j \leq n_1 + \ldots + n_i$ *and* $i = 1, \ldots, s-1$. *Then* $R((B_1, \ldots, B_s)\ A\,; x_1, \ldots, x_s))$ *is a rational function of* $x_1, \ldots, x_s$ *over the field* $\boldsymbol{Q}$ *of rational numbers. Further its denominator is*

$$\prod_{k=1}^{s} \prod_{i=0}^{m_k} (1 - p^{(n_1+\ldots+n_k-i)(-m+n+i+1)}(x_1\ldots x_k)^2) \prod_{k=1}^{s} (1 - x_1\ldots x_k)^{m'_k},$$

where $m'_k = 1$ *or* $= 0$ *according as* $r(A) \geq n_1 + \ldots + n_k$ *or not. In particular if* $m \geq 2n+2$, *the denominator of the above power series is*

$$\prod_{k=1}^{s} \prod_{i=0}^{n_1+\ldots+n_k-1} (1 - p^{(n_1+\ldots+n_k-i)(-m+n+i+1)}(x_1\ldots x_k)^2) \prod_{k=1}^{s} (1 - x_1\ldots x_k).$$

Remark 1. In [1], for non-degenerate symmetric matrices $B_1, \ldots, B_s$, and A with entries in $\boldsymbol{Z}_p$, Böcherer and Sato defined a formal power series $Q((B_1, \ldots, B_s), A\,; x_1, \ldots, x_s)$ by

$$Q((B_1, \ldots, B_s), A\,; x_1, \ldots, x_s) = \sum_{e_1, \ldots, e_s=0}^{\infty} \alpha_p(\mathrm{diag}(p^{e_1} B_1, \ldots, p^{e_s} B_s), A) x_1^{e_1} \ldots x_s^{e_s},$$

and showed that it is a rational function of $x_1, \ldots, x_s$ over $\boldsymbol{Q}$. On the other hand, we define a formal power series $P((B_1, \ldots, B_s), A\,; x_1, \ldots, x_s)$ by

$$P((B_1, \ldots, B_s), A\,; x_1, \ldots, x_s) = \sum_{e_1, \ldots, e_s=0}^{\infty} \alpha_p(\mathrm{diag}(p^{2e_1} B_1, \ldots, p^{2e_s} B_s), A) x_1^{e_1} \ldots x_s^{e_s},$$

which is a special case of the one defined in [3]. As stated in [3] and [4], the above two types of power series are related with each other. In [4], we obtained an explicit form of the denominator of $P((B_1, \ldots, B_s), A\,; x_1, \ldots, x_s)$, and therefore, of $Q((B_1, \ldots, B_s), A\,; x_1, \ldots, x_s)$ when $n_1 = \ldots = n_s = 1$ and $p \neq 2$. On the other hand, as easily seen, $Q((B_1, \ldots, B_s), A\,; x_1, \ldots, x_s)$ can be expressed as a $\boldsymbol{Q}[x_1, \ldots, x_s]$-linear combination of several power series defined in this note. For example, if $b_1, b_2 \in \boldsymbol{Z}_p \setminus \{0\}$, we have

$$Q((b_1, b_2), A\,; x_1, x_2) = R((b_1, b_2), A\,; x_1, x_2) + R((b_2, b_1), A\,; x_1, x_2) - R(\mathrm{diag}(b_1, b_2), A\,; x_1 x_2).$$

Thus, by Theorem 3, we can also obtain an explicit form of the denominator of $Q((B_1, \ldots, B_s), A\,; x_1, \ldots, x_s)$, and therefore of $P((B_1, \ldots, B_s), A\,; x_1, \ldots, x_s)$ when A is even integral unimodular and $B_1, \ldots, B_s$ are diagonal, which will appear elsewhere.

Remark 2. By the above theorem we see that the denominator of $R(B, A\,; x)$ is

$$\prod_{i=0}^{\min(n-1, r(A))} (1 - p^{(n-i)(-m+n+i+1)} x^2)(1 - x)^{m'},$$

where $m' = 1$ or $= 0$ according as $r(A) \geq n$ or not. This is a refinement of the result of [2], [6].

Remark 3. Theorem 3 can be generalized to the case where A is an arbitrary non-degenerate matrix if $p \neq 2$.

Remark 4. In the above results, the condition that B_i are diagonal is not necessary if $p \neq 2$.

Now we show that our result on the denominator of the above power series is best possible by giving a simple example. Let $p \neq 2$, $m = 3$, $n = 2$, and $n_1 = n_2 = 1$. Let A be a unimodular symmetric matrix of degree 3 with entries in $\boldsymbol{Z}_p$ and b_1, b_2 be elements of the group $\boldsymbol{Z}_p^*$ of p-adic units. We assume that $\chi(b_1 \det A) = 1$ and $\chi(-b_2 \det A) = -1$, where χ is the quadratic residue symbol defined modulo p. Then by [5] we have

$$R((b_1, b_2), A; x_1, x_2) = \frac{(1-p^{-2})(1+2x_1^2x_2+x_1^2x_2^2)}{(1-x_1^2)(1-x_1^2x_2^2)(1-px_1^2x_2^2)}.$$

Thus the reduced denominator of $R((b_1, b_2), A; x_1, x_2)$ is $(1-x_1^2)(1-x_1^2x_2^2)(1-px_1^2x_2^2)$. We note that $r(A) = 1$, and therefore $m_1 = 0$ and $m_2 = 1$. Thus Theorem 3 is best possible.

Acknowledgement. The author would like to thank the referee for many valuable comments.

References

[1] S. Böcherer and F. Sato: Rationality of certain formal power series related to local densities. Comment. Math. Univ. St. Paul., **36**, 53–86 (1987).

[2] Y. Hironaka: On a denominator of Kitaoka's formal power series attached to local densities. ibid., **37**, 159–171 (1988).

[3] H. Katsurada: Generalized Igusa local zeta functions and local densities of quadratic forms. Tôhoku Math. J., **44**, 211–218 (1992).

[4] ——: A certain formal power series of several variables attached to local densities of quadratic forms. I (to appear in J. Number Theory).

[5] Y. Kitaoka: A note on local densities of quadratic forms. Nagoya Math. J., **92**, 145–152 (1983).

[6] ——: Local densities of quadratic forms and Fourier coefficients of Eisenstein series. ibid., **103**, 149–160 (1986).