44. *A Note on the Mean Value of the Zeta and L-functions. VIII*

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The aim of the present note is to show an idea that may perhaps lead us to a clue to higher power moments of the Riemann zeta-function. In particular we shall indicate that the eighth power moment problem may be reduced to the theory of automorphic forms over the full modular group.

Thus let us first put

$$I(T, G; p) = \frac{1}{G\sqrt{\pi}} \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + i(T+t)\right) \right|^{2p} e^{-(t/G)^2} dt$$
$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + i(T+Gt)\right) \right|^{2p} e^{-t^2} dt,$$

where p is a positive integer, and G, T are arbitrary positive parameters. Our problem is to find a good upper bound or more preferably an asymptotic formula for $I(T, G; p)$ as T tends to infinity; in these the lower bound of G is another important issue. So far, only the initial two cases $p = 1, 2$ have been successfully investigated; the first case is due to Atkinson [1] and the second to the present author [4][5] (see also Ivić's lecture notes [2]). We are now concerned with the case $p \geq 3$ where no satisfactory results have been obtained yet.

We start our discussion with a trivial observation: The Cauchy-Schwarz inequality gives

(1) $$I(T, G; 2p) \geq (I(T, G; p))^2.$$

We then pose the question: Is it possible to correct this crude inequality to an identity? Our idea is to appeal to the theory of orthogonal polynomials in order to answer this question.

For this sake let

(2) $$\varphi_n(t) = (2^n n! \sqrt{\pi})^{-1/2} e^{-t^2/2} H_n(t), \quad n = 0,1,2\ldots,$$

where

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n e^{-t^2}}{dt^n}$$

is the nth Hermite polynomial. The set $\{\varphi_n\}$ forms a complete orthonormal system of the Hilbert space $L^2(-\infty, \infty)$ (see e.g., Lebedev [3, Chap. 4]). We thus have the Fourier-Hermite expansion

(3) $$\pi^{-1/4} \left| \zeta\left(\frac{1}{2} + i(T+Gt)\right) \right|^{2p} e^{-t^2/2} = \sum_{n=0}^{\infty} z_n(T, G; p) \varphi_n(t),$$

where

(4) $$z_n(T, G; p) = \pi^{-1/4} \int_{-\infty}^{\infty} \varphi_n(t) \left| \zeta\left(\frac{1}{2} + i(T+Gt)\right) \right|^{2p} e^{-t^2/2} dt.$$

The Parseval formula gives the identity

$$I(T, G; 2p) = \sum_{n=0}^{\infty} z_n(T, G; p)^2. \tag{5}$$

We note that $z_0(T, G; p) = I(T, G; p)$. Hence the inequality (1) is a consequence of (5), or we may say that the latter is a result of correcting the former to an identity.

Next we invoke the expansion

$$e^{2t\xi-\xi^2} = \sum_{n=0}^{\infty} \frac{H_n(t)}{n!} \xi^n.$$

The definition (2) transforms this into

$$e^{-(t-\xi)^2+t^2/2} = \pi^{1/4} \sum_{n=0}^{\infty} \frac{\varphi_n(t)}{\sqrt{n!}} \left(\frac{\xi}{\sqrt{2}}\right)^2.$$

Hence we have, by (4),

$$\begin{aligned} I(T+G\xi, G; p) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + i(T+Gt)\right) \right|^{2p} e^{-(t-\xi)^2} dt \\ &= \sum_{n=0}^{\infty} \frac{z_n(T, G; p)}{\sqrt{n!}} \left(\frac{\xi}{\sqrt{2}}\right)^n. \end{aligned} \tag{6}$$

This implies that in (5) we may compute each $z_n(T, G; p)$ in terms of the Taylor expansion of $I(T+G\xi, G; p)$ with respect to ξ. We remark also that (3) may be taken for a means of relating individual values of the zeta-function with the mean values.

So far we have not used any particular properties of the zeta-function. In fact, as is easily seen, the above argument can be developed for any continuous functions of polynomial order, say, in place of $\left|\zeta\left(\frac{1}{2} + i(T+Gt)\right)\right|^{2p}$. But, we are now taking into account of the special fact that for $p = 1, 2$ there exist explicit formulas for $I(T, G; p)$. A remarkable property shared by those formulas is that the variables T and G appear explicitly and are well separated. Hence $z_n(T, G; p)$, $p = 1,2$, can be computed effectively via the relation (6). We shall make the situation specific in the case $p = 2$ only, for there we may have a possibility of leaping from the fourth power moment to the eighth power moment of the zeta-function that has defied all attempts to solve.

To this end we define some standard symbols from the theory of automorphic forms: We denote by $\left\{\lambda_j = \kappa_j^2 + \frac{1}{4}; \kappa_j > 0, j = 1,2, \ldots\right\} \cup \{0\}$ the discrete spectrum of the hyperbolic Laplacian acting on the space of all non-holomorphic automorphic functions with respect to the full modular group. Let ψ_j be the Maass wave form correspoding to λ_j so that $\{\psi_i\}$ forms an orthonormal base of the space spanned by all cusp forms, and each ψ_j is an eigen-function of every Hecke operator $T(n)$ $(n \geq 1)$. Thus there exists a certain real number $t_j(n)$ such that $T(n)\psi_j = t_j(n)\psi_j$. The Hecke series attached to ψ_j is defined by

$$\sum_{n=1}^{\infty} t_j(n) n^{-s}, \quad (\mathrm{Re}(s) > 1),$$

which converges to an entire function denoted by $\mathscr{H}_j(s)$. As for the holomorphic cusp forms, we let $\{\psi_{j,k}\,;\,1 \le j \le \vartheta(k)\}$ stand for the orthonormal base, consisting of eigen functions of all Hecke operators $T_k(n)$, of the Petersson unitary space of holomorphic cusp forms of weight $2k \ge 12$ with respect to the full modular group. This means, in particular, that we have $T_k(n)\psi_{j,k} = t_{j,k}(n)\psi_{j,k}$ with a certain real number $t_{j,k}(n)$. As before we introduce the Hecke series

$$\sum_{n=1}^{\infty} t_{j,k}(n)n^{-s}, \quad (\mathrm{Re}(s) > 1).$$

Again this converges to an entire function, which we denote by $\mathscr{H}_{j,k}(s)$. Finally, with the first Fourier coefficients ρ_j of ψ_j and $\rho_{j,k}$ of $\psi_{j,k}$ we put $\alpha_j = |\rho_j|^2(\cosh \pi\kappa_j)^{-1}$ and $\alpha_{j,k} = 16(2k-1)!(4\pi)^{-2k-1}|\rho_{j,k}|^2$, respectively.

Then we have the explicit formula ([5, Theorem] with a minor modification): For any positive G and T,

$$\begin{aligned} I(T, G\,;2) &= M(T, G) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\left|\zeta\left(\frac{1}{2}+ir\right)\right|^6}{|\zeta(1+2ir)|^2}\Theta(r\,;T, G)dt \\ &\quad + \sum_{j=1}^{\infty}\alpha_j\mathscr{H}_j\left(\frac{1}{2}\right)^3\Theta(\kappa_j\,;T, G) + \sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}\mathscr{H}_{j,k}\left(\frac{1}{2}\right)^3\Xi\left(k-\frac{1}{2}\,;T, G\right). \end{aligned} \tag{7}$$

Here

$$\Theta(r\,;T, G) = \mathrm{Re}\left\{\left(1+\frac{i}{\sinh(\pi r)}\right)\Xi(ir\,;T, G)\right\}\,;$$

$$\begin{aligned} \Xi(\eta\,;T, G) &= \frac{\Gamma\left(\frac{1}{2}+\eta\right)^2}{\Gamma(1+2\eta)}\int_0^{\infty} x^{-1-\eta}(1+x)^{-\frac{1}{2}}\cos\left(T\log\left(1+\frac{1}{x}\right)\right) \\ &\quad \times F\left(\frac{1}{2}+\eta, \frac{1}{2}+\eta\,;1+2\eta\,;-\frac{1}{x}\right)\exp\left(-\left(\frac{G}{2}\log\left(1+\frac{1}{x}\right)\right)^2\right)dx, \end{aligned}$$

where F is the hypergeometric function. The term $M(T, G)$ is of simple character; if $G < T(\log T)^{-1}$, say, it is essentially equal to a polynomial of fourth degree in $\log T$.

Now the combination of (6) and (7) gives immediately

$$\begin{aligned} z_n(T, G\,;2) &= M^{(n)}(T, G) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\left|\zeta\left(\frac{1}{2}+ir\right)\right|^6}{|\zeta(1+2ir)|^2}\Theta^{(n)}(r\,;T, G)dt \\ &\quad + \sum_{j=1}^{\infty}\alpha_j\mathscr{H}_j\left(\frac{1}{2}\right)^3\Theta^{(n)}(\kappa_j\,;T, G) \\ &\quad + \sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}\mathscr{H}_{j,k}\left(\frac{1}{2}\right)^3\Xi^{(n)}\left(k-\frac{1}{2}\,;T, G\right), \end{aligned} \tag{8}$$

where

$$\Theta^{(n)}(r\,;T, G) = \mathrm{Re}\left\{\left(1+\frac{i}{\sinh(\pi r)}\right)\Xi^{(n)}(ir\,;T, G)\right\}\,;$$

$$(9)\quad \Xi^{(n)}(\eta\,;T,G)=\frac{(\sqrt{2}G)^n\Gamma\left(\frac{1}{2}+\eta\right)^2}{\sqrt{n!}\,\Gamma(1+2\eta)}\int_0^\infty x^{-1-\eta}(1+x)^{-\frac{1}{2}}$$
$$\times\left(\log\left(1+\frac{1}{x}\right)\right)^n\exp\left(-\left(\frac{G}{2}\log\left(1+\frac{1}{x}\right)\right)^2\right)$$
$$\times\cos\left(T\log\left(1+\frac{1}{x}\right)+\frac{n}{2}\pi\right)F\left(\frac{1}{2}+\eta,\frac{1}{2}+\eta\,;1+2\eta\,;-\frac{1}{x}\right)dx.$$

The term $M^{(n)}(T,G)$ can be computed by expanding $M(T+G\xi,G)$ with respect to ξ.

Inserting (8) into (5) we conclude, therefore, that the eighth power moment of the zeta-function is *explicitly* expressible in terms of the objects in the theory of automorphic forms over the full modular group. The actual value of this explicit formula will, however, become clear only after further investigations such as how to effectively truncate the series in (5) and estimate $z_n(T,G\,;2)$ via (9) for relevant values of n.

Remark. One may also consider an application of the Parseval formula to the Fourier integral

$$\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+i(T+Gt)\right)\right|^{2p}e^{-t^2/2+i\xi t}dt.$$

But this approach takes the matter to a complicated situation relating to the behavior of the zeta-function off the critical line. An advantage of the above argument is that we may focus our attention on the critical line.

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References

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