39. Einstein Metrics on a Product Manifold

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Abstract: We get that a necessary and sufficient condition for the product manifold (M, g) of two compact connected irreducible Riemannian symmetric spaces to be an Einstein manifold is that the metric g is a critical point for the total scalar curvature function $I(g)$ defined in the set $\mathscr{A}(M)$ of some proper volume preserving Riemannian metrics, and if g is critical for $I(g)$, then g gives minimum of $I(g)$.

1. Introduction and statement of result. Let us consider the set $\mathscr{M}^1(M)$ of Riemannian metrics g on a given compact orientable C^∞ manifold M such that

$$\int_M dV_g = 1,$$

where dV_g is the volume element of (M, g). It is well known that a metric $g \in \mathscr{M}^1(M)$ is an Einstein metric if and only if g is critical for total scalar curvature

$$S(g) = \int_M s(g)\, dV_g$$

in $\mathscr{M}^1(M)$, where $s(g)$ is the scalar curvature of (M, g).

In this paper, we treat the case of a product manifold $(M, g) = (M_1 \times M_2, g_1 \times g_2)$ of two compact connected irreducible Riemannian symmetric spaces (M_1, g_1) and (M_2, g_2) of dimension m_1 and m_2 respectively. In general, the product manifold of two Einstein manifolds is not Einstein. We denote by $\mathscr{A}(M)$, the set of all Riemannian metrics g on the product manifold $(M, g) = (M_1 \times M_2, g_1 \times g_2)$ satisfying the following conditions:

(C.1) (M_1, g_1) and (M_2, g_2) are compact connected irreducible Riemannian symmetric spaces,

and

(C.2) the volume of product manifold $(M, g) = (M_1 \times M_2, g_1 \times g_2)$ is 1.

We now consider the function

$$I(g) = \int_M s(g)\, dV_g,$$

defined on $\mathscr{A}(M)$. One can ask for the critical points of the function on $\mathscr{A}(M)$ (cf. [2]). Simultaneously a question then arises that whether a critical point g gives a minimum of $I(g)$ or not (cf. [2, 3, 4]).

We introduce a well known Lemma.

Lemma A. *Let V be an n-dimensional real vector space, Γ a connected Lie subgroup of $GL(V)$. Assume ϕ and ψ be Γ-invariant definite quadratic forms on*

V. Then, if (Γ, V) is an irreducible representation, $\phi = c\psi$ for some proper non zero real constant c.

In this paper, we get the following main result.

Theorem. *Let (M, g) be the product manifold of two compact connected irreducible Riemannian symmetric spaces (M_1, g_1) and (M_2, g_2) such that $g \in \mathscr{A}(M)$. Then the metric $g \in \mathscr{A}(M)$ on M is an Einstein metric iff the metric g is a critical of $I(g)$ in $\mathscr{A}(M)$.*

Moreover, if a metric $g(\in \mathscr{A}(M))$ is a critical point of $I(g)$ in $\mathscr{A}(M)$, then g gives a minimum of $I(g)$ for any deformation $g(t)$ of g in $\mathscr{A}(M)$.

§2. Proof of the main Theorem. An irreducible Riemannian symmetric space is an Einstein manifold (cf. [1, Corollary 7.74, p. 194]). Hence (M_1, g_1) and (M_2, g_2) are Einstein manifolds. We put $g_1 =: h_1$ and $g_2 =: h_2$ such that $Vol(M_1, g_1) = Vol(M_2, g_2) = 1$. Then, $h_1 \times h_2 =: h \in \mathscr{A}(M)$. We denote by s_1 (resp. s_2), the scalar curvature of (M_1, h_1) (resp. (M_2, h_2)).

Because of isotropy irreducibility, we get the following from Lemma A: *for an arbitrary given metric $g = g_1 \times g_2$ belonging to $\mathscr{A}(M)$, there exist constant positive real numbers c_1 and c_2 such that $g_1 = c_1 h_1$ and $g_2 = c_2 h_2$.* Since $Vol(M, g) = 1$, $dim(M_1) = m_1$ and $dim(M_2) = m_2$, we have

$$c_2 = c_1^{-\frac{m_1}{m_2}}. \tag{1}$$

Moreover, (M_1, h_1) (resp. (M_2, h_2)) has constant scalar curvature s_1 (resp. s_2). Hence (M_1, g_1) (resp. (M_2, g_2)) has constant scalar curvature $\frac{s_1}{c_1}$ (resp. $c_1^{\frac{m_1}{m_2}} s_2$). For an arbitrary given smooth deformation $g(t) = g_1(t) \times g_2(t)$ of $g = g_1 \times g_2 = ch_1 \times c^{-\frac{m_1}{m_2}} h_2$ in $\mathscr{A}(M)$ such that $g(0) = g$, we can put $g_1(t) = c_1(t) h_1$ and $g_2(t) = c_1(t)^{-\frac{m_1}{m_2}} h_2$, where $c_1(t)$ is a positive valued smooth function satisfying $c_1(0) = c_1$. $(M_1, g_1(t))$ (resp. $(M_2, g_2(t))$ has scalar curvature $\frac{s_1}{c_1(t)}$ $\left(\text{resp. } \frac{s_2}{c_2(t)}\right)$. We immediately obtain

$$\frac{d}{dt} dV_{g(t)} = 0, \tag{2}$$

$$\frac{d}{dt} I(g(t)) = \int_M \left(\frac{-s_1}{c_1(t)^2} + \frac{m_1}{m_2} c_1(t)^{\frac{m_1 - m_2}{m_2}} s_2 \right) c_1'(t)\, dV_{g(t)}, \tag{3}$$

$$\frac{d}{dt} I(g(t)) \Big|_{t=0} = \int_M \left(-\frac{s_1}{(c_1)^2} + \frac{m_1}{m_2} c_1^{\frac{m_1 - m_2}{m_2}} s_2 \right) c_1'(0)\, dV_g. \tag{4}$$

Hence, g is a critical point of $I(g)$ in $\mathscr{A}(M)$ iff

$$\frac{s_1}{(c_1)^2} = \frac{m_1}{m_2} c_1^{\frac{m_1 - m_2}{m_2}} s_2, \tag{5}$$

i.e.,

$$\frac{s_1}{c_1 m_1} = \frac{s_2}{c_1^{-\frac{m_1}{m_2}} m_2} = \frac{s_2}{c_2 m_2}.$$

Thus, $g \in \mathscr{A}(M)$ is a critical point of $I(g)$ if and only if (M, g) is an Einstein manifold.

Finally, differentiating (3) again we get

(6)

$$\frac{d^2}{dt^2} I(g(t)) = \int_M \left[\begin{array}{l} \left(\frac{2s_1}{c_1(t)^3} + \frac{m_1(m_1 - m_2)}{(m_2)^2} c_1(t)^{\frac{m_1-2m_2}{m_2}} s_2\right) c_1'(t)^2 \\ \quad + \left(\frac{m_1}{m_2} c_1(t)^{\frac{m_1-m_2}{m_2}} s_2 - \frac{s_1}{c_1(t)^2}\right) c_1''(t) \end{array} \right] dV_{g(t)}.$$

Now, if $g(0)$ is an Einstein metric, then we have from (5) and (6)

$$(7) \qquad \frac{d^2}{dt^2} I(g(t)) \,|_{t=0} = \int_M \left(\frac{(m_1 + m_2)s_1}{(c_1)^3 m_2} c_1'(0)^2\right) dV_g.$$

Thus, every Einstein metric g in $\mathscr{A}(M)$ gives a minimum of $I(g)$.

References

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