38. *Index for Factors Generated by Direct Sums of II_1 Factors*

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In this paper, we give an index formula for II_1 factors generated by increasing sequences of infinite dimensional algebras and some examples of such factors. The theory in case of finite dimensional algebras was constructed by H. Wenzl.

§1. Preliminaries. Let $M=\oplus_{j=1}^m M_j$ be a finite direct sum of II_1 factors and q_j the minimal central projection corresponding to M_j. Since the normalized normal trace on a II_1 factor is unique, a trace on M (denoted by tr) is decided by a numerical vector $\vec{s}=(tr(q_i))_{i=1,\dots,m}$, called the trace vector of M. Let $N=\oplus_{i=1}^n N_i\subset M$ be an another finite direct sum of II_1 factors and p_i the corresponding minimal central projection. We assume that the trace on N is the restriction of the trace on M, and denote by $\vec{t}$ the trace vector of N.

We define two matrices relating the inclusion relation $N\subset M$, the index matrix and the trace matrix. The index matrix $\Lambda_N^M=(\lambda_{ij})$ is given by

$$\lambda_{ij}=\begin{cases}[M_{p_iq_j}:N_{p_iq_j}]^{1/2} & p_iq_j\neq 0,\\ 0 & p_iq_j=0,\end{cases}$$

and the trace matrix $T_N^M=(t_{ij})$ by $t_{ij}=tr_{M_j}(p_iq_j)$, where tr_{M_j} is the unique normalized normal trace on M_j.

We suppose that N is of finite index in M, i.e., there is a faithful representation π of M on a Hilbert space such that $\pi(N)'$ is finite. Then the algebra $\langle M, e_N\rangle$ obtained by basic construction for $N\subset M$ is a finite direct sum of II_1 factors and the corresponding minimal central projections are $Jq_1J,\dots,Jq_mJ$, where J is the canonical conjugation on $L^2(M, tr)$. We know the following in [1].

(1.1) The equality $\vec{t}=T_N^M\,\vec{s}$ holds.

(1.2) $$\Lambda_M^{\langle M,e_N\rangle}=(\Lambda_N^M)^t$$

(1.3) $$T_M^{\langle M,e_N\rangle}=\tilde{T}_N^M F_N^M,$$

where $(\tilde{T}_N^M)_{ji}=\begin{cases}\dfrac{\lambda_{ij}^2}{t_{ij}} & p_iq_j\neq 0\\ 0 & p_iq_j=0,\end{cases}$, $F_N^M=diag(\varphi_1,\cdots,\varphi_n)$, $\varphi_i=(\sum_j(\tilde{T}_N^M)_{ji})^{-1}$.

(1.4) For any trace Tr on $\langle M, e_N\rangle$, $Tr(e_Np_i)=\varphi_iTr(Jp_iJ)$.

The index $[M:N]$ is defined as $[M:N]=r(\tilde{T}_N^MT_N^M)$, where $r(T)$ is the spectral radius of T.

Now let $M_0\subset M_1$ be a pair of II_1 factors with finite index and trivial relative commutant. By the basic construction, we obtain a tower of II_1 factors $M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$. Since the relative commutant $M'_{n-1} \cap M_n$ is trivial, we have in this case:

(1.5) $$tr_{M_n}(x) = tr_{M'_0}(x) \text{ for } x \in M'_0 \cap M_n.$$

§2. Factors generated by direct sums of II_1 factors. We construct a pair of factors from finite direct sums of II_1 factors and calculate the index for the pair.

Lemma 1. *Let $N \subset M$ be a pair of II_1 von Neumann algebras acting on a Hilbert space H with finite dimensional centers. Let tr be a faithful finite trace on M and E_N the trace preserving conditional expectation of M onto N. Suppose a projection $e \in B(H)$ satisfies $exe = E_N(x)e$ for all $x \in M$ and $eN \cong N$. Then,*

(1) *$\langle M, e\rangle = A \oplus B$, with $A \cong \langle M, e_N\rangle$, and $B \cong C$ uw-closed subalgebra of M.*

(2) *Let $z \in \langle M, e\rangle$ be the central projection onto A. Then z is equal to the central support of e.*

(3) *Let Tr be a trace on $\langle M, e\rangle$ such that $Tr|_M = tr$, then*

$$Tr(e) \geq d\cdot Tr(z), \text{ where } d = min\{\varphi_i = (F_N^M)_{ii}\,;\, i = 1,\dots,n\}.$$

Proof. (1) Let M_1 be a $*$-algebra generated by $M \cup \{e\}$, and define an ultrastrong ($=$ us) continuous $*$-homomorphism $\Phi : M_1 \rightarrow \langle M, e_N\rangle$ by $\Phi(x_0 + \sum_{i=1}^n x_i e y_i) = x_0 + \sum_{i=1}^n x_i e_N y_i$, and denote the extension of Φ to $\langle M, e\rangle$ by φ. Then the map φ is ultraweak ($=$ uw) continuous $*$-homomorphism from $\langle M, e\rangle$ onto $\langle M, e_N\rangle$. Put $B = Ker(\varphi) \subset \langle M, e\rangle$, then B is a uw-closed two-sided ideal of $\langle M, e\rangle$ and there exists a central projection $z \in \langle M, e\rangle$ such that $B = (1-z)\langle M, e\rangle$. Define $A = z\langle M, e\rangle$, then $\varphi : A \rightarrow \langle M, e_N\rangle$ is a $*$-isomorphism. Therefore

$$\langle M, e\rangle = A \oplus B \text{ and } A \cong \langle M, e_N\rangle.$$

(2) The proof is simple and so we omit it.

(3) Let $\{p_i\}_{i=1}^n$ be the minimal central projections of N and $\{\tilde{p}_i\}_{i=1}^n$ the corresponding central projections of $A \cong \langle M, e_N\rangle$. Moreover let $\Psi : A \rightarrow \langle M, e_N\rangle$ be a $*$-isomorphism such that $\Psi(\tilde{p}_i) = Jp_iJ$, where J is as above. Take another trace $Tr' = Tr\circ\Psi^{-1}$ on $\langle M, e\rangle$, then

$$Tr(e\tilde{p}_i) = Tr'(e_N Jp_iJ) = \varphi_i Tr'(Jp_iJ) = \varphi_i Tr(\tilde{p}_i) \geq d\cdot Tr(\tilde{p}_i),$$

and therefore $Tr(e) = \sum_i Tr(e\tilde{p}_i) \geq d\sum_i Tr(\tilde{p}_i) = d\cdot Tr(z)$. **Q.E.D**

Let $\{M_n\}_{n\in \boldsymbol{N}}$ and $\{N_n\}_{n\in \boldsymbol{N}}$ be two increasing sequences of direct sums of II_1 factors such that, for each $n \in \boldsymbol{N}$, the following is a commuting square:

$$\begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1}. \end{array}$$

Here we treat two conditions.

Condition I (Periodicity). There exist $n_0 \geq 1$ and $p \geq 1$ such that for any $n \geq n_0$, $T_{N_n}^{N_{n+1}}$, $T_{M_n}^{M_{n+1}}$ and $F_{N_n}^{M_n}$ are periodic with period p and $T_{N_n}^{N_{n+p}}$, $T_{N_n}^{N_{n+p}}$ are primitive.

Condition II (Lower Boundedness). There exists a constant $d > 0$ such that $(F_{N_n}^{M_n})_{ii} \geq d$ for all n and i.

It is clear that Condition II follows from Condition I.

We put $M = (\cup M_n)''$ and $N = (\cup N_n)''$.

Lemma 2. *Let $\{M_n\}_{n\in \boldsymbol{N}}$ and $\{N_n\}_{n\in \boldsymbol{N}}$ are as above.*

(1) *If Condition I holds, M and N are II_1 factors.*

(2) *If Condition II holds and M, N are II_1 factors, then $[M : N] < \infty$.*

Proof. (1) Let tr be a normalized trace on M and $\vec{s}_n$ the trace vector of tr for M_n. We may suppose that $n_0 = p = 1$. Then putting $T_{M_n}^{M_{n+1}} = T$ for any $n \in \boldsymbol{N}$, we have by (1.1),

$$\vec{s}_n = T^k \vec{s}_{n+k} \text{ for all } k \geq 1.$$

So $\vec{s}_n \in \cap_k T^k(\boldsymbol{R}^+)^m$, where $\boldsymbol{R}^+ = \{x \in \boldsymbol{R} \,;\, x > 0\}$, i.e., $\vec{s}_n$ is a Perron Frobenius eigenvector of T. Therefore the normalized trace on M is unique so that M is a II_1 factor.

(2) Let z_n be the central support of e_N in $\langle M_n, e_N\rangle$, then $z_n \to 1$ (us). Take a semifinite trace Tr on $\langle M, e_N\rangle$. Since $e_N\langle M, e_N\rangle e_N = Ne_N \cong N$, we see that e_N is a finite projection and $Tr(e_N) < \infty$. From Lemma 1 (3), we get $Tr(e_N) \geq d\cdot Tr(z_n)$ for all $n \in \boldsymbol{N}$, and letting $n \to \infty$,

$$Tr(e_N) \geq d\cdot Tr(1) \quad \text{or} \quad Tr(1) \leq d^{-1}Tr(e_N) < \infty.$$

Therefore $\langle M, e_N\rangle$ is finite so that $[M : N]$ is finite. **Q.E.D**

Now we give a new index formula which is one of our main results.

Theorem 1. *Let $\{M_n\}_{n\in \boldsymbol{N}}$ and $\{N_n\}_{n\in \boldsymbol{N}}$ are as above.*

(1) *Assume M and N are II_1 factors, and $[M : N] < \infty$. Then*

$$[M : N] = \lim_n \langle \vec{t}_n, \vec{f}_n\rangle,$$

where $\vec{f}_n = ((F_{N_n}^{M_n})_{ii}^{-1})_i$, $\vec{t}_n$ is the trace vector of N_n and $\langle \cdot , \cdot \rangle$ is the standard inner product.

(2) *If Condition I holds, then for all $n \geq n_0$*

$$[M : N] = \langle \vec{t}_n, \vec{f}_n\rangle = [M_n : N_n].$$

Proof. (1) Since the index $[M : N]$ is finite, there exists a normalized trace tr on $\langle M, e_N\rangle$ such that

$$tr(xe_N) = [M : N]^{-1}tr(x) \text{ for } x \in M.$$

Using Lemma 1, we get a uw-closed subalgebra A of $\langle M_n, e_N\rangle$, $*$-isomorphic to $\langle M_n, e_{N_n}\rangle$. Let $\{p_i\}_{i=1}^m$ be the minimal central projections of N_n, $\{\tilde{p}_i\}_{i=1}^m$ be the corresponding central projections of A, and $\Psi : A \to \langle M_n, e_{N_n}\rangle$ be the $*$-isomorphism such that $\Psi(\tilde{p}_i) = Jp_iJ$, where J is the canonical conjugation on $L^2(M_n, e_{N_n})$ is the canonical conjugation. Take the trace $tr' = tr\circ\Psi^{-1}$ on $\langle M_n, e_{N_n}\rangle$, then

$$tr(\tilde{p}_i) = tr'(Jp_iJ) = \varphi_{n,i}^{-1}\, tr'(e_{N_n}p_i) = \varphi_{n,i}^{-1}\, tr(e_N p_i) = \varphi_{n,i}^{-1}\, [M : N]^{-1}tr(p_i),$$

where $\varphi_{n,i} = (F_{N_n}^{M_n})_{ii}$. Denoting the trace vector of N_n for tr by $\vec{t}_n = (t_{n,1}, \ldots, t_{n,m})$, ($m$ depends on n) and the central support of e_N for $\langle M_n, e_N\rangle$ by z_n, we get

$$tr(z_n) = \sum_i \varphi_{n,i}^{-1}[M : N]^{-1}tr(p_i) = [M : N]^{-1}\sum_i \varphi_{n,i}^{-1}t_{n,i}$$
$$= [M : N]^{-1}\langle \vec{t}_n, \vec{f}_n\rangle.$$

Since $z_n \to 1$ (uw) as $n \to \infty$, it follows that $\lim_n\langle \vec{t}_n, \vec{f}_n\rangle = [M : N]$.

(2) If Condition I holds, then for $n \geq n_0$ the trace vector $\vec{t}_n$ is a Perron Frobenius eigenvector of $S_n = T_{N_n}^{N_{n+p}}$ by the proof of Lemma 2. Since S_n and $F_{N_n}^{M_n}$ are periodic with period p, $\vec{t}_n$ and $\vec{f}_n$ are also periodic for $n \geq n_0$. Be-

cause $\langle \vec{t}_n, \vec{f}_n \rangle$ converges to $[M : N]$, we have for $n \geq n_0$

$$[M : N] = \langle \vec{t}_n, \vec{f}_n \rangle \text{ and } z_n = 1.$$

From $z_n = 1$, we have $\langle M_n, e_{N_n} \rangle \cong \langle M_n, e_N \rangle$, and so there exists a $*$-isomorphism $\Psi : \langle M_n, e_{N_n} \rangle \to \langle M_n, e_N \rangle$. Let tr be a Markov trace on $\langle M_n, e_N \rangle$, then $tr' = tr \circ \Psi$ is also a Markov trace on $\langle M_n, e_{N_n} \rangle$. Let $\vec{s}_n$ be the trace vector of M_n, then

$$\tilde{T}_n T_n \vec{s}_n = [M : N]\, \vec{s}_n,$$

where $T_n = T_{N_n}^{M_n}$ and $\tilde{T}_n = \tilde{T}_{N_n}^{M_n}$. Therefore $\vec{s}_n$ is a Perron Forbenius eigenvector of $\tilde{T}_n T_n$ so that

$$[M : N] = r(\tilde{T}_n T_n) = [M_n : N_n].$$

Q.E.D

Remark. In case that M_n and N_n are finite direct sums of full matrix algebras, the same formula holds too. This formula is different from Wenzl's index formula in [4], but essentially the same.

§3. Examples. We give examples of $\{M_n\}_n$ and $\{N_n\}_n$ satisfying Condition II.

Let $A_{-1} \subset A_0$ be an irreducible pair of II_1 factors with index λ. If $\lambda < 4$, there exists $k \in \boldsymbol{N}$ such that $\lambda = 4\cos^2(\pi/k)$. In case $\lambda \geq 4$, we put $k = \infty$. By the basic construction we get a sequences of II_1 factors $A_{-1} \subset A_0 \subset A_1 = \langle A_0, e_1 \rangle \subset A_2 = \langle A_1, e_2 \rangle \subset \cdots$, where $e_i = e_{A_{i-2}}$. Now we define $N_0 = A_0$, $N_i = \langle A_{-1}, e_1, \ldots, e_i \rangle$ for $i \geq 1$ and $M_j = A_j$ for $j \geq 0$. Then $N_n \cong N \otimes \langle e_1, \ldots, e_n \rangle$, so we can see the structure of N_n from that of $\langle e_1, \ldots, e_n \rangle$. This fact is important in the sequel.

Lemma 3. *For all n, the following is a commuting square*:

$$\begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1}. \end{array}$$

Next we calculate the matrices $T_{M_n}^{M_{n+1}}$, $T_{N_n}^{N_{n+1}}$ and $T_{N_n}^{M_n}$. First it is clear that $T_{M_n}^{M_{n+1}} = (1)$.

Proposition 1. *Let $\Lambda_{N_n}^{N_{n+1}}$ be the index matrix and $T_{N_n}^{N_{n+1}}$ the trace matrix of the inclusion $N_n \subset N_{n+1}$. Then,*

$$\Lambda_{N_n}^{N_{n+1}} = (d_{i,j}^{(n)})_{ij},\ d_{i,j}^{(n)} = \begin{cases} 1 & j = i,\ i+1, \\ 0 & \text{otherwise}, \end{cases}$$

$$T_{N_n}^{N_{n+1}} = (c_{i,j}^{(n)})_{ij},\ c_{i,j}^{(n)} = \begin{cases} \dfrac{\alpha_{n,i}}{\alpha_{n+1,j}} & j = i,\ i+1, \\ 0 & \text{otherwise}, \end{cases}$$

where, for $n \leq k-3$,

$$i = 0,1,\ldots,\left[\frac{n+1}{2}\right],\ j = 0,1,\ldots,\left[\frac{n+2}{2}\right],\ \alpha_{n,j} = \binom{n}{i} - \binom{n}{i-2},$$

and for $n \geq k-2$,

$$i = \left[\frac{n-k+4}{2}\right],\ldots,\left[\frac{n+1}{2}\right],\ j = \left[\frac{n-k+5}{2}\right],\ldots,\left[\frac{n+2}{2}\right],$$

$$\alpha_{n,j} = \binom{n}{i} - \binom{n}{i-2} - \binom{n}{i+k-2}.$$

We can prove this proposition by induction on n using Lemma 1.

Proposition 2. *Let $\Lambda_{N_n}^{M_n}$ be the index matrix and $T_{N_n}^{M_n}$ the trace matrix of the inclusion $N_n \subset M_n$. Then,*

$$T_{N_n}^{M_n} = (c_i^{(n)}) \text{ with } c_i^{(n)} = \alpha_{n,i}\lambda^{-i}P_{n+2-2i}(\lambda^{-1}),$$
$$\Lambda_{N_n}^{M_n} = (d_i^{(n)}) \text{ with } d_i^{(n)} = \lambda^{\frac{n+1-2i}{2}}P_{n+2-2i}(\lambda^{-1}),$$

where $i = 0, \dots, \left[\dfrac{n+1}{2}\right]$ *for* $n \leq k-3$; $i = \left[\dfrac{n-k+4}{2}\right], \dots, \left[\dfrac{n+1}{2}\right]$ *for* $n \geq k-2$, *and* $\alpha_{n,i}$ *is the constant in the previous proposition, and* $P_n(t)$ *is Jones polynomial defined by* $P_0(t) = P_1(t) = 1$ *and* $P_n(t) = P_{n-1}(t) - tP_{n-2}(t)$.

Proof. Let $\{p_{n,i}\}_i$ be the minimal central projections corresponding to the factorization of N_n. Since $T_{N_n}^{M_n} = (tr(p_{n,i}))_i$, it is easy to see that $c_i^{(n)} = \alpha_{n,i}\lambda^{-i}P_{n+2-2i}(\lambda^{-1})$. We prove the assertion for $\Lambda_{N_n}^{M_n}$ by induction on n. Since $d_0^{(0)} = [A_0 : A_{-1}]^{1/2} = \lambda^{1/2}$, the statement is clear for $n = 0$. Suppose it is true for $n = m$. For $j = i, i+1$,

$$\begin{aligned}(d_j^{(m+1)})^2 &= [(M_{m+1})_{p_{m+1,j}} : (N_{m+1})_{p_{m+1,j}}] \\ &= [(M_{m+1})_{p_{m+1,j}p_{m,i}} : (N_{m+1})_{p_{m+1,j}p_{m,i}}] \\ &= [(M_{m+1})_{p_{m+1,j}p_{m,i}} : (N_m)_{p_{m+1,j}p_{m,i}}] \\ &= tr_{(M_{m+1})_{p_{m,i}}}(q)\, tr_{(N_m)'_{p_{m,i}}}(q)\, [(M_{m+1})_{p_{m,i}} : (N_m)_{p_{m,i}}],\end{aligned}$$

where $q = p_{m+1,j}p_{m,i}$.

Denote by tr the trace on M_{m+1}, then $tr_{(M_{m+1})_{p_{m,i}}}(q) = tr(p_{m,i})^{-1}tr(q)$ and $tr_{(N_m)'_{p_{m,i}}}(q) = tr_{N_0'}(p_{m,i})^{-1}tr_{N_0'}(q) = tr(p_{m,i})^{-1}tr(q)$ by (1.5). So

$$\begin{aligned}(d_j^{(m+1)})^2 &= tr(p_{m,i})^{-2}tr(q)^2[(M_{m+1})_{p_{m,i}} : (M_m)_{p_{m,i}}][(M_m)_{p_{m,i}} : (N_m)_{p_{m,i}}] \\ &= tr(p_{m,i})^{-2}tr(p_{m+1,j})^2 tr_{(N_{m+1})_{p_{m+1,j}}}(q)^2\,(d_i^{(m)})^2\lambda.\end{aligned}$$

Using the hypothesis of induction and Proposition 1, we obtain

$$\begin{aligned}d_j^{(m+1)} &= \frac{\alpha_{m,i}}{\alpha_{m+1,j}}\, tr(p_{m,i})^{-1}tr(p_{m+1,j})\,d_i^{(m)}\,\lambda^{1/2} \\ &= \lambda^{(m+2-2j)/2}P_{m+3-2i}(\lambda^{-1}).\end{aligned}$$

Q.E.D

Put $M = (\cup_n M_n)''$ and $N = (\cup_n N_n)''$, then M and N are II_1 factors (cf. [3]).

Theorem 2. *Let $A_{-1} \subset A_0$ be an irreducible pair of II_1 factors with index λ and construct $\{M_n\}_n$ and $\{N_n\}_n$ by the above method.*

(1) *$\{M_n\}_n$ and $\{N_n\}_n$ satisfy* Condition *II if and only if the index $\lambda < 4$.*

(2) *The index $[M : N]$ is given by*

$$[M : N] = \begin{cases} \dfrac{k}{4\sin^2\dfrac{\pi}{k}} & \lambda < 4, \\ \infty & \lambda \geq 4, \end{cases}$$

where k is an integer such that $\lambda = 4\cos^2(\pi/k)$.

Proof. Let $\{p_{n,i}\}_i$ be the minimal central projections corresponding to the factorization of N_n. Then the trace vector $\vec{t}_n$ of N_n is equal to $(tr(p_{n,i}))_i$ and the vector $\vec{f}_n = (f_{n,i})_i$ in Theorem 1 is given by $\vec{f}_n = (tr(p_{n,i})^{-1}(d_i^{(n)})^2)_i$ with $d_i^{(n)} = [(M_n)_{p_{n,i}} : (N_n)_{p_{n,i}}]^{1/2}$. By Proposition 2,

$$(f_{n,i})^{-1} = \alpha_{n,i}/(\lambda^{n+1-2i} P_{n+2-2i}(\lambda^{-1})).$$

a) Case of $\lambda < 4$: Since $P_n((4\cos^2\theta_k)^{-1}) = \sin n\theta_k/(2^{n-1}\cos^{n-1}\theta_k \sin\theta_k)$ with $\theta_k = \pi/k$,

$$(f_{n,i})^{-1} = \alpha_{n,i}2^{n+1-i}\sin\theta_k/(\sin(n+2-2i)\theta_k\cos^{n+1}\theta_k) \geq \sin\theta_k.$$

Therefore we see that Condition II holds. Further by Theorem 1,

$$[M:N] = \lim_n \langle \vec{t}_n, \vec{f}_n\rangle = \lim_n \sum_{i=[(n-k+4)/2]}^{[(n+1)/2]} tr(p_{n,i})tr(p_{n,i})^{-1}(d_i^{(n)})^2$$

$$= \lim_n \sum_{i=[(n-k+4)/2]}^{[(n+1)/2]} \frac{\sin^2(n+2-2i)\theta_k}{\sin^2\theta_k} = \frac{k}{4\sin^2\frac{\pi}{k}}.$$

b) Case of $\lambda \geq 4$: By simple calculation, it follows that

$$(f_{n,0})^{-1} = \alpha_{n,0}/(\lambda^{n+1}P_{n+2}(\lambda^{-1})) \leq \lambda^{-n/2} \to 0 \ (n \to \infty).$$

So, Condition II doesn't hold. Now suppose that $[M:N] < \infty$. Then by Theorem 1,

$$[M:N] = \lim_n \langle \vec{t}_n, \vec{f}_n\rangle = \lim_n \sum_{i=0}^{[(n+1)/2]} (d_i^{(n)})^2$$

$$= \lim_n \sum_{i=0}^{[(n+1)/2]} \lambda^{n+1-2j} P^2_{n+2-2j}(\lambda^{-1}) \geq \lim_n \sum_{i=0}^{[(n+1)/2]} \lambda^{-1} = \infty.$$

This is a contradiction, so that $[M:N] = \infty$.

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