5. Graded Algebras of Vector Bundle Maps over an Elliptic Curve

By Daisuke TAMBARA

Department of Mathematics, Hirosaki University

(Communicated by Heisuke HIRONAKA, M. J. A., Jan 12, 1994)

We study here a kind of homogeneous coordinate rings of matrix algebras over an elliptic curve. Let X be an elliptic curve over an algebraically closed field k with $\mathrm{char}(k) \neq 2$. Choose a point $P \in X$ and let $\mathscr{L} = \mathscr{L}(P)$ be the invertible $\mathscr{O}_X$-module associated to the divisor P. For a positive integer n let $\mathscr{E}_n$ be an indecomposable locally free $\mathscr{O}_X$-module of rank n which is a successive extension of $\mathscr{O}_X$. Such a module exists uniquely up to isomorphism ([2]). We form the $\mathscr{O}_X$-algebra $\mathscr{E}nd(\mathscr{E}_n)$, the sheaf of local endomorphisms of $\mathscr{E}_n$, and then form a graded k-algebra

$$\Lambda(n) = \bigoplus_{i \geq 0} \Gamma(X, \mathscr{E}nd(\mathscr{E}_n) \otimes \mathscr{L}^{\otimes i}) = \bigoplus_{i \geq 0} \mathrm{Hom}(\mathscr{E}_n, \mathscr{E}_n \otimes \mathscr{L}^{\otimes i}).$$

In this paper we give an explicit description of the algebra $\Lambda(n)$. Details and proofs will appear elsewhere.

1. Realization of $\Lambda(n)$ as a matrix algebra. Put $S = \bigoplus_{i \geq 0} \Gamma(X, \mathscr{L}^{\otimes i})$. This is a commutative graded k-algebra. For an $\mathscr{O}_X$-module $\mathscr{F}$ we put $\Gamma_*(\mathscr{F}) = \bigoplus_{i \in \mathbf{Z}} \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes i})$, which is a graded S-module. Also $\Lambda(n)$ is an S-algebra. Since $\mathscr{L}$ is ample, we have $\Lambda(n) \cong \mathrm{End}_S(\Gamma_*(\mathscr{E}_n))$ as S-algebras (cf. [1]).

The algebra S is generated by suitable homogeneous elements t, x, y of degree 1, 2, 3, respectively, with relation $y^2 = x(x - t^2)(x - \lambda t^2)$ for some $\lambda \in k - \{0, 1\}$ ([3, p. 336]). We fix t, x, y, λ throughout. Put $v = x - (\lambda + 1)t^2$, $u = (x - t^2)(x - \lambda t^2)$.

Let $R = k[t, x]$, a polynomial subalgebra of S. Then $S = R \oplus Ry$. Define a graded S-module M as follows. M is a free graded R-module with basis $\alpha, \beta_i, \gamma_i$ for $i > 0$ with $\deg \alpha = 0, \deg \beta_i = 1, \deg \gamma_i = 2$. The action of y on M is given by

$$y\alpha = x\beta_1 + t\gamma_1$$
$$y\beta_i = -\lambda t^3 O_i \beta_{i-1} - tx\beta_{i+1} + v\gamma_{i-1} - t^2\gamma_{i+1}$$
$$y\gamma_i = x^2\beta_{i+1} + \lambda t^3 E_i \gamma_{i-1} + tx\gamma_{i+1}$$

where $\beta_0 = -t\alpha$, $\gamma_0 = x\alpha$ and $O_i = 1$ for an odd i, $O_i = 0$ for an even i, $E_i = 1 - O_i$. For $n \geq 1$ define a graded S-submodule $M(n)$ of M to be the free R-submodule generated by $\alpha, \beta_i, \gamma_i$ for $1 \leq i \leq n - 1$ and $x\beta_n + t\gamma_n$.

Proposition 1. *$\Gamma_*(\mathscr{E}_n) \cong M(n)$ as graded S-modules.*

So we can identify $\Lambda(n) = \mathrm{End}_S(M(n))$.

Though the S-module M is not free, the $S\left[\frac{1}{y}\right]$-module $M\left[\frac{1}{y}\right] = S\left[\frac{1}{y}\right] \otimes_S M$ is free with basis $\alpha_i, i \geq 0$, given by $\alpha_i = \frac{1}{x}\gamma_i$ if i is odd, $\alpha_i =$

$-\frac{1}{u}(\lambda t^3\beta_i - v\gamma_i)$ if i is even. Also $M(n)\left[\frac{1}{y}\right]$ has a basis α_i for $0 \leq i \leq n - 1$.

2. Generators, relations and bases. We first give generators of $\Lambda = \Lambda(n)$. Define an $S\left[\frac{1}{y}\right]$-linear map $f : M(n)\left[\frac{1}{y}\right] \to M(n)\left[\frac{1}{y}\right]$ by

$$f(\alpha_i) = \alpha_{i-1} - \frac{\lambda t^3 y}{ux}\alpha_{i-2} + \frac{((\lambda+1)v + \lambda t^2)x}{u}\alpha_{i-3} - \frac{\lambda ty}{u}\alpha_{i-4} + \frac{\lambda vx}{u}\alpha_{i-5} \qquad \text{if } i \text{ is even}$$

$$f(\alpha_i) = \alpha_{i-1} + \frac{\lambda t^3 y}{ux}\alpha_{i-2} + \frac{(\lambda+1)x - \lambda t^2}{x}\alpha_{i-3} + \frac{\lambda ty}{u}\alpha_{i-4} \qquad \text{if } i \text{ is odd}$$

where we understand $\alpha_i = 0$ for $i < 0$. It can be shown that f restricts to an S-linear map $M(n) \to M(n)$ of degree 0, which we denote also by f. We have $f^n = 0$ and the degree 0 part Λ_0 of Λ is an n dimensional k-algebra generated by f.

We can also define an S-linear map $g : M(n) \to M(n)$ as follows. When n is even,

$$g(\alpha_0) = t\alpha_{n-1} - \frac{y}{x}\alpha_{n-2}$$

$$g(\alpha_1) = \frac{y}{x}\alpha_{n-1} + \frac{t((\lambda+1)x - \lambda t^2)}{x}\alpha_{n-2} + \frac{\lambda t^2 y}{u}\alpha_{n-3}$$

$$g(\alpha_2) = -\frac{\lambda t^2 y}{u}\alpha_{n-2} + \frac{\lambda tvx}{u}\alpha_{n-3}$$

$$g(\alpha_i) = 0 \text{ for } i > 2,$$

and when n is odd,

$$g(\alpha_0) = t\alpha_{n-1} - \frac{vy}{u}\alpha_{n-2}$$

$$g(\alpha_1) = \frac{y}{x}\alpha_{n-1} + (\lambda+1)t\alpha_{n-2}$$

$$g(\alpha_2) = -\frac{\lambda t^2 y}{u}\alpha_{n-2} + \sum_{i \geq 3, \text{odd}} \lambda(-\lambda-1)^{(i-3)/2}\left(t\alpha_{n-i} - \frac{vy}{u}\alpha_{n-i-1}\right)$$

$$g(\alpha_i) = 0 \text{ for } i > 2.$$

Then g is a map of degree 1, so belongs to the degree 1 part Λ_1.

From now on we assume $n > 2$.

Theorem 2. *Λ is a free R-module of rank $2n^2$ with basis $f^i, f^i g f^j, f^i g f^{n-3} g f^j, f^i g f^{n-2} g f^{n-3} g$ for $0 \leq i \leq n-1$, $0 \leq j \leq n-2$.*

Regard Λ as a left $\Lambda_0 \otimes \Lambda_0$-module by $(a \otimes b) \cdot \phi = a\phi b$.

Theorem 3. *$\Lambda_+ = \bigoplus_{i>0}\Lambda_i$ is a free $\Lambda_0 \otimes \Lambda_0$-module with basis $(gf^{n-1})^i g$, $(gf^{n-1})^i (gf^{n-2})^j gf^{n-3} g$ for $i, j \geq 0$.*

Theorem 4. *The k-algebra Λ is generated by f and g. The relations between them are generated by the following ones.*

Case n is even: $f^n = 0$ *and* $n-2$ *quadratic relations of the form*

$$gf^k g = A_k \cdot gf^{n-3}g + B_k \cdot gf^{n-1}g \text{ for } 0 \leq k \leq n-2,\ k \neq n-3$$

with $A_k, B_k \in \Lambda_0 \otimes \Lambda_0$.

Case n is odd: $f^n = 0$ *and* $n-2$ *quadratic relations as above and one cubic relation of the form*

$$gf^{n-3}gf^{n-3}g = C \cdot gf^{n-2}gf^{n-3}g + D \cdot gf^{n-1}gf^{n-3}g + E \cdot gf^{n-1}gf^{n-1}g$$

with $C, D, E \in \Lambda_0 \otimes \Lambda_0$.

The theorems fail when $n = 2$. The generators of $\Lambda(2)$ should be f, g, h, where h is an element of degree 2 defined by $h(\alpha_0) = x\alpha_1$, $h(\alpha_1) = 0$.

3. Case n is even. The relations in the previous theorem are implicit, but when n is even, we can give explicit defining equations for Λ, using additional generators. We define $e \in \Lambda_0$ and $g_+ \in \Lambda_1$ by

$$e(\alpha_i) = \alpha_{i-2} \text{ for all } i$$

$$g_+(\alpha_0) = t\alpha_{n-2} - \frac{vy}{u}\alpha_{n-3}$$

$$g_+(\alpha_1) = t\alpha_{n-1} + (\lambda + 1)t\alpha_{n-3}$$

$$g_+(\alpha_2) = \frac{vy}{u}\alpha_{n-1} + (\lambda + 1)t\alpha_{n-2}$$

$$g_+(\alpha_i) = 0 \text{ for } i > 2.$$

Theorem 5. *If n is even and $n > 2$, the k-algebra Λ has the following presentation. The generators are f, e, g, g_+. The relations are*

$$e^{\frac{n}{2}} = 0$$

$$f^2 = (1 + (\lambda + 1)e)(1 + \lambda e)(1 + e)e$$

$$fg(1 + (\lambda + 1)e) + (1 + (\lambda + 1)e)gf$$

$$= g_+ + (\lambda + 1)eg_+ + (\lambda + 1)g_+e + \lambda e^2 g_+ + ((\lambda + 1)^2 + \lambda)eg_+e + \lambda g_+e^2 + \lambda(\lambda + 1)e^2 g_+ e + \lambda(\lambda + 1)eg_+e^2$$

$$ge^{\frac{n-4}{2}}g = \lambda g_+ e^{\frac{n-2}{2}} g_+$$

$$g_+ e^{\frac{n-4}{2}} g_+ = (\lambda + 1) g_+ e^{\frac{n-2}{2}} g_+$$

$$ge^j g = ge^j g_+ = 0 \text{ for } 0 \leq j \leq \frac{n-6}{2}.$$

References

[1] M. Artin and M. Van den Bergh: Twisted homogeneous coordinate rings. J. Algebra, **133**, 249–271 (1990).

[2] M. F. Atiyah: Vector bundles over an elliptic curve. Proc. London Math. Soc., **7**, 414–452 (1957).

[3] R. Hartshorne: Algebraic Geometry. Springer-Verlag, New York (1977).